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AURORAL PARALLACTIC PHOTOGRAPHY

PART I
by
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PARTS 2 and 3
by
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PREFACE

The three parts of this issue were written as self-contained papers, for separate publication, each complete within its own subject. A decision was made, however, to publish them together and with little alteration so that any one part could be read alone. A certain amount of repetition, without which the clarity of any single part would be impaired, was thus necessarily retained, e.g. the basic auroral problem which is the subject of Part 2 is also outlined briefly in the Introduction to Part 1 and in Section VII of Part 3.

Part 1 describes the design of the instrument for measuring auroral photographs while Part 3, dealing with its use, describes the actual method proposed and forms thus the main part of the whole subject of auroral parallactic photography.

References from one part to another were kept to a minimum and were generally intended to serve the purpose of greater detail rather than that of clarification.

G.V.S.

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Errata

p. 9 line 10 from bottom: for $\frac{t_2}{n}$ read $\frac{t_2}{n}$

p. 22 line 22: for "than" read "then".

p. 23 line 2: in the expression "$\rho_\lambda \cos \phi$" inverted gamma printed for subscript lambda.

p. 51 line 5 from bottom: for "theodolites" read "theodolite".

p. 61 line 7: for "presented" read "present".

p. 67 lines 9 and 8 from bottom: for ". . . and be restricted to . . ." read ". . . can be restricted to . . ."

AURORAL PARALLACTIC PHOTOGRAPHY

PART 1

INSTRUMENT FOR MEASURING PARALLACTIC PHOTOGRAPHS

By

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(Manuscript received 14 April 1964)

ABSTRACT

A collimator and theodolite combination gives direct readings of spherical co-ordinates for the points on a photographic plate which is located in the focal plane of the collimator. A graticule inside the collimator facilitates the re-identification of auroral points.

CONTENTS

I. INTRODUCTION

The instrument described below was designed for measurements of auroral parallactic plates taken for the purpose of determination of auroral position in space, viz. height above sea level and distance from the observer.

The solution is basically a surveyor's problem but with a twofold difference. Firstly, the aurora moves making visual measurements from two bases not feasible. Photographs taken simultaneously (in pairs) from the two bases fix the aurora as it was located at that moment and allow the necessary time for measurements. Secondly, there are, as a rule, no identifiable discrete points on the aurora which could be selected on both photographs. This difficulty is overcome by a parallactic law which states that a point at finite distance measured on the photograph taken at the first base will be parallactically displaced on the photograph taken at the second base along a specific line, the line of parallactic displacement, whose orientation on the plate can be calculated.

To make the problem clearer let us suppose that a star is seen exactly at the lower auroral border on the first photograph and we select that point to be measured. On the second photograph that star, being at infinity, gives us an identical pointing in space. The aurora, being at finite distance, is parallactically displaced and appears on the second plate well off the star. We can fit to the star the line of parallactic displacement whose orientation on the second plate can be computed. The intersection of the line with the lower auroral border on the second photograph will re-identify the point selected on the first photograph because the parallactic displacement of the selected point "proceeded" along the line, beginning with the star and ending with the point of intersection. The location of the auroral point on the second photograph can now be measured and a combination of this measurement with the one on the first photograph allows a determination of distance and height of the aurora to be tackled as a surveyor's problem. There is of course no need for a star to be actually present. Any pointing on the first photograph on an arbitrarily selected auroral point can be immediately transformed to a corresponding point at infinity on the second photograph and the line of parallactic displacement "fitted" to it as described above with respect to the star.

The auroral problem was discussed by Störmer (1955) and Harang (1951).

Jacka and Ballantyne (1955) have described phototheodolites which were used for securing auroral photographs. These cameras were provided with a focal plane graticule and graduated circles giving azimuth and elevation. The impression of the graticule appeared on the photographs and the circles indicated the centre-of-graticule co-ordinates from which the plates could be measured.

II. THE COLLIMATING PRINCIPLE

Unless a special device is made, the points on a photographic plate can be measured only in linear co-ordinates, x and y, selecting any convenient unit of length. Transformation of linear to spherical co-ordinates (zenith distance and azimuth) takes up a large part of the total time required for the complete auroral solution. The device evolved to avoid the transformation of the co-ordinates is the auroral collimator. It works on the same principle as the goniometric method of lens calibration which gives a correlation between the linear and angular measurement in the focal plane. If a developed photographic plate is put back into the camera with which it was taken and viewed through the object glass, the angular relations on the plate are restored and the measurement can be made with a theodolite placed in front of the camera objective. In order to have a correct orientation as well, up side up, the whole camera must be turned 180° around its optical axis because the original photograph was reversed. Right and left are then still interchanged, unless a positive contact print of the plate is made, but this aspect gives no trouble during measurement.

The importance of turning the whole camera instead of the plate in the camera, when highest precision is required, lies in the irregularities of the field characteristics (scale) of the objective which depend on direction as well as distance from the optical axis. In other words, the position of the objective in the camera must not be altered. Due to features of the design of the auroral collimator the above point had to be ignored but an accurate lens calibration, made at Defence Standards Laboratory, Melbourne, had shown an extreme possible error of $\frac{3}{4}$ of a minute of arc within 40° field and generally not exceeding half a minute which could be tolerated for the purpose in hand. Nevertheless, extensive test measures were carried out to ascertain that greater errors do not occur.

Camera lenses used at the two bases for photographing auroras were Leitz Summarit f/1·5, 5 cm focal length. As no adaptor was made for interchanging the lenses in the auroral collimator only one of them was used as the collimating lens and slight corrections, derived from lens calibration, were in some cases applied to the measurements of plates which were exposed with the other lens. Generally, for lenses with identical specifications, such corrections would not be necessary.

III. DESCRIPTION OF THE INSTRUMENT

The photographic plate is illuminated by a collimated beam from a condenser consisting of two identical 32 mm plano-convex lenses each of 85 mm focal length (44 mm equivalent focal length in combination). Comparatively long focal length and unidirectional orientation of the lenses (convex faces towards the photographic plate) guard against excessive spherical aberration which has proven detrimental to the contrast of the auroral image. The lens of the auroral collimator focuses the condenser beam on the entrance pupil of the theodolite.

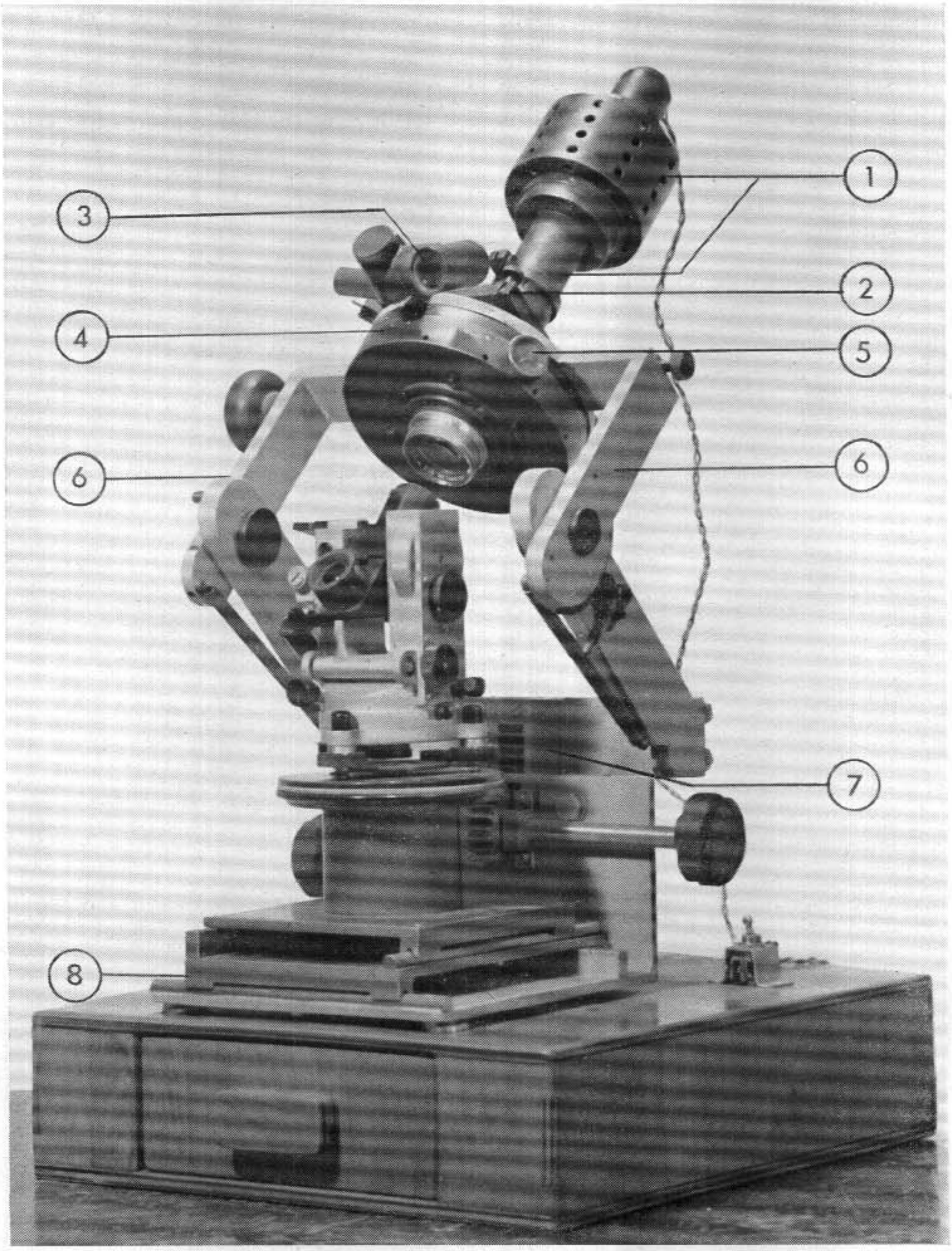

FIG. 1. General View of the Instrument.

1. Condenser; 2. Plate Holder; 3. Vernier Magnifier; 4. Graduated Rotating Drum; 5. Milled Screw Head for lateral movement of parallactic graticule; 6. Trunnion Arms; 7. Vertical Slide; 8. Base with Horizontal Slides.

The general view of the instrument is shown on Figure 1. All parts are made of blackened brass with the exception of the base with the movable platform which is made of mild steel. A cylindrical tube at the top with circumferential holes for heat dissipation holds the lamp, heat filter and diffuser. A smaller diameter extension of this tube holds condenser lenses which are permanently focused on the diffuser and is a

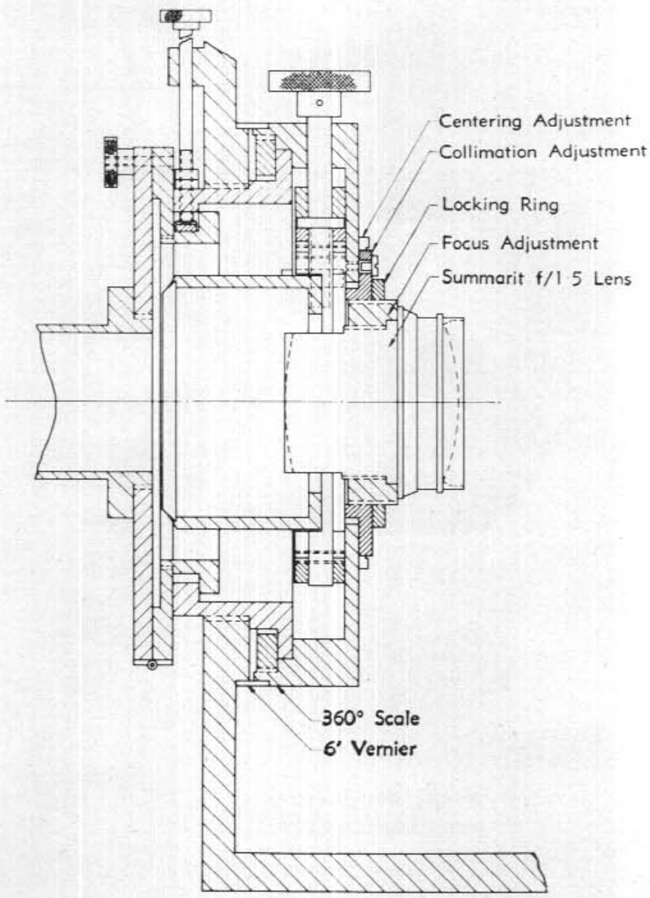

FIG. 2. Auroral Collimator (side view).

sliding fit in the tube attached to the plate holder. The lenses, diffuser, and heat filter are held by means of spacers and bezel rings. The sliding of the two tubes serves for adjustment of the spacing between the condenser and the photographic plate so as to preclude any direct visibility of the condenser lenses in the theodolite. This visibility can be tested by putting a coloured thread across the condenser lens.

Figures 2 and 3 show the collimator and its plate holder. The latter is made of two brass plates hinged at one end and held in the closed position by a screw locking device at the other. The front plate is recessed and fitted with leaf springs and pressure pads to hold the negative firmly in position. The unit has a cylindrical flanged extension which operates in the main body allowing it to rotate freely about the optic axis. Attached to the main body and maintaining pressure on this extension are two screws at 90° to one another with a spring loaded plunger diametrically

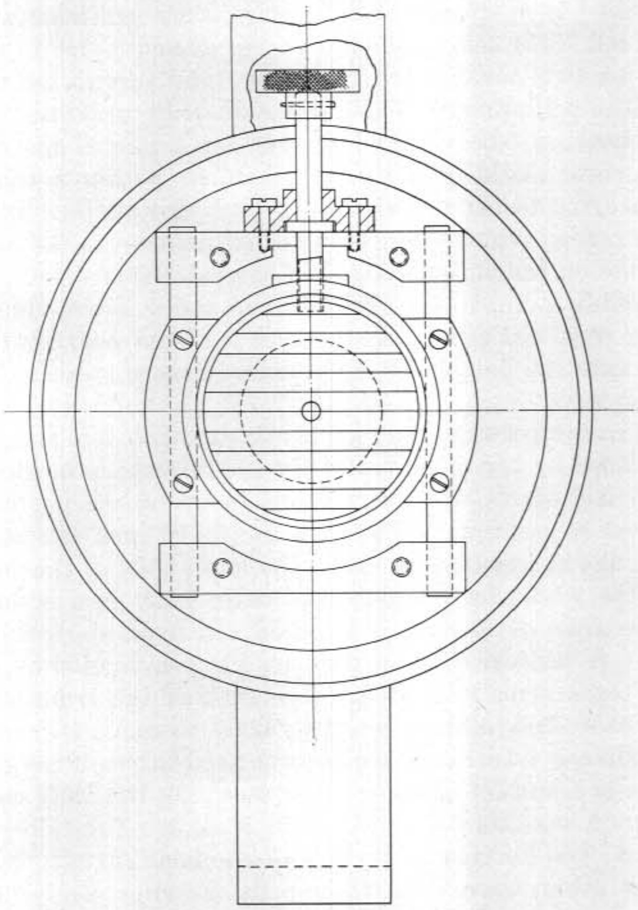

FIG. 3. Auroral Collimator (end view).

opposed to a position between the screws. This movement permits an accurate centring of the photographic plate. Three small annular gibs are used as pads at the ends of the screws and plunger allowing the plate holder to rotate freely.

The collimating lens is screwed into a brass cell which is in turn screwed into a flanged cell fitted to the rotating drum of the collimator by three radial screws through oversize holes in the flange. This double cell construction allows the lens to be adjusted longitudinally for focus (collimation) and locked with a screwed ring, then radially for centering on the axis of rotation of the drum.

The rotating drum carrying a 360° scale is a friction movement fit in the main body and has holes in the circumference permitting insertion of a rod for leverage fine setting. The vernier scale is attached to the main body and the readings are made through a fixed illuminated magnifier.

The lines of parallactic displacement, mentioned in the introduction, are ruled (with $\frac{1}{4}$ inch spacing) on the parallactic graticule which is spun into a brass cell. This cell is flanged at the opposite end to the graticule and fitted with two parallel steel rods by four screws, allowance being made for initial adjustment. The steel rods with graticule cell attached slide in two bearing blocks thus allowing for a lateral movement of the graticule, at right angle to the graticule lines, within a range of about $\frac{1}{3}$ inch. The bearing blocks are held to the anterior surface of the rotating drum by four screws which permit a small adjustment in the final centering of the graticule on the optical axis. The lateral displacement of the graticule is achieved by a nut in the graticule cell and a screw, running parallel to the sliding rods and extending through a thrust bearing in one of the blocks to the external periphery of the drum. A zero position is marked on the thimble on the screw.

The separation between the parallactic graticule and the photographic plate is controlled by the thickness of the pressure pads in the plate holder already described above. With the optics in use a spacing of 0·002 inch produces as yet no perceptible parallax and 0·004 inch can be tolerated.

Figure 1 shows in sufficient detail the main body of the collimator and the base of the whole instrument. The main body is a screwed annulus with trunnion arms diametrically opposed and fixed at right angles to its plane surface. A vertical slide on the base has two angled extension arms. The ends of these arms are fitted to the ends of the trunnion arms with bearing journals. This allows the main body to move through a 90° arc and be locked in any position by a quadrant fixed to one of the arms.

The base is a section of right angle steel. On the back vertical face a rack and pinion operated dovetail slide is fixed. This slide carries the extended arms. The horizontal surface of the base carries two tables, one on top of the other, each on a three-point bearing slide and moving at right angles to each other. On the top table the Askania theodolite is fixed. The horizontal and vertical movements serve the purpose of alignment between the collimating lens and theodolite in any pointing of the latter and they do not affect the pointing itself owing to the collimated light used (object at infinity).

IV. ALIGNMENT BETWEEN THEODOLITE AND COLLIMATOR

Each photographic plate bears an impression of the focal plane graticule with which the cameras (photo theodolites) were provided and which is called the photographic graticule to distinguish it from the parallactic graticule inside the collimator. The photographic graticule indicates the vertical orientation (vertical lines) and the centre, the co-ordinates of which are recorded on the photographic plate.

As a preliminary to alignment the parallactic graticule is set at zero position with its lines in vertical orientation with respect to the theodolite. Using the lateral movement and the rotation of the plateholder the photographic plate is aligned to the parallactic graticule. The centre of the photographic graticule is then on the optical axis of the collimating lens and its vertical alignment agrees with that of the theodolite.

The theodolite is set on the reading of zenith distance as recorded on the photographic plate and the centre of the photographic graticule is brought into coincidence with the pointing cross of the theodolite by rotating the trunnion arms of the collimator. The horizontal rotation of the theodolite must be used in this alignment after which the horizontal circle is set on zero or on the azimuth recorded on the photographic plate, depending on whether the negative or the positive plates are used.

With the collimator aligned to the theodolite the latter reads the spherical co-ordinates of points on the photographic plate. Conversely, its pointing cross gives a visible indication of the location of a point the co-ordinates of which were computed and set on the circles of the theodolite. A line of parallactic displacement can be set through the computed point by means of drum rotation and lateral movement of the parallactic graticule.

The theodolite does not need to be "levelled". In the correct alignment between auroral collimator and theodolite the latter is, in fact, levelled to the horizon of the photographic plate whatever its orientation is with respect to the horizon of the laboratory in which it is used. The levelling screws are, however, retained on the theodolite because they are useful in its fine alignment to the collimator.

V. EYEPIECE DESIGN

Askania Midget Theodolite, Type Tkmi, was chosen, mainly because of its small size. It gives fast and easy readings, as is usual with most modern theodolites, with a precision of about 0·1 minute of arc which is better than actually required in conjunction with the auroral collimator. With the eyepiece supplied it gave 16 times magnification which is about 10 times more than the acceptable limit and a new eyepiece had to be designed.

Viewing the collimated auroral photograph through a theodolite is the same as viewing the actual aurora through the same theodolite and one would naturally wish to have a low magnification, even no magnification at all. The magnification was set to be between one and two coupled with large true field of not less than 20° and with large eye relief which is explained below.

A low magnification eyepiece is much larger than the standard one and in order to fit it within the fork mounting of the theodolite a diagonal eyepiece, with aluminized mirror, was designed. It has an additional advantage of convenience. Figure 4 shows the eyepiece and also the mounting of the objective in two eccentric rings for centering. The eyepiece is a modified Ramsden type with field lens **fixed** in the focal plane giving full utilization of the true field which is only restricted by the inner diameter of the theodolite tube. It also makes possible the ruling of two cross lines, for pointings, directly on the flat side of the field lens. Focusing the eyepiece is by means of variation of spacing between the field and the eye lenses, which was found to work well within a considerable range without affecting the aberrations.

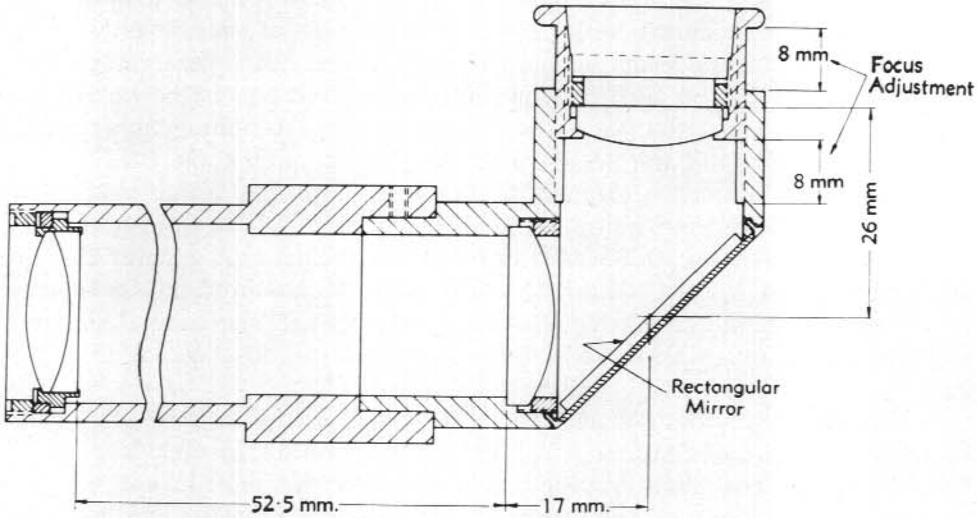

FIG. 4. Eyepiece.

An advantage of the Ramsden eyepiece lies in good optical performance apart from lateral colour which is harmless when viewing black and white photographs with low illumination and when there is an easy possibility to use filters as well. This feature is coupled with comparative ease of optical design, using largely Conrady's (1957) theory. There is, however, for the designer an unpleasant deficiency in the procedure, namely that the eyepiece is not fitted, designwise, to the available telescope, but a telescope must be fitted to the designed eyepiece. The simplicity of designing lies exactly in this reversed procedure but at the initial stages the designer is completely in the dark, as only ray tracing results will enable him to compute the focal length of the telescope with which the eyepiece will perform well. It is actually not the focal length but the "Tube Length", as Conrady (loc. cit.) calls it. It may be expressed as the optical distance between the focal plane and the exit pupil of the objective fitted with a stop located well in front. Such a Conrady stop can easily be fitted to the available telescope of the theodolite. Conrady's statement (loc. cit.) on the insensitivity of the characteristics of the Ramsden eyepiece to the "Tube Length" applies rather to moderate magnifications only. For extremely low magnification, as in the present case, the performance of the eyepiece in conjunction with the theodolite was found to suffer if the calculated "Tube Length" was greater, even by only 50 per cent, than the focal length of the theodolite which is about $54\frac{1}{2}$mm. No suitable set of field and eye lens combination was found which would give less than 80 mm "Tube Length" and a value of 89 mm was accepted in conjunction with the use of a Conrady stop, placed at about 18 mm in front of the objective. The stop, not shown on Figures 1 and 4, is located at the end of a tube which is a sliding fit to the telescope of the theodolite.

The third condition quoted above was that the eyepiece must have a great eye relief, i.e. the distance between the eye lens and the eye point (located at the exit pupil of the telescope). The value of the eye relief is greatly reduced by the use of the Conrady stop, and the reduced value must remain large enough to overcome the deep location of the eye lens in its mounting which was necessary for the focusing arrangement.

Combinations of field and eye lenses satisfying the three conditions of field, magnification and eye relief were found by slide-ruling paraxial relations which is much faster than ray tracing.

The eye relief (E) can be computed from:

$$E = f_e\left(\frac{M+1}{M} - \frac{d}{f_1}\right) - \frac{t_2}{n}$$

Here f_e is the effective focal length of the eyepiece,

$M = \frac{f_o}{f_e}$ the magnification that the eyepiece gives with the telescope of focal length f_o (theodolite),

d the spacing between the field and the eye lenses,

f_1 the focal length of the field lens,

t_2 the thickness of the eye lens, and

n the refractive index of glass.

Without the last term the eye relief would be reckoned from the second principal point of the eye lens and the expression $\frac{t_2}{n}$ is correct only for a plano-convex lens.

Suitable sets were then subjected to the longer process of the ray tracing which gives the "Tube Length" and the aberrational characteristics.

The final choice fell on a set with components of equal focal length of 36 mm spaced 31·5 mm apart, giving the equivalent focal length of the eyepiece 32 mm, magnification 1·7, and the "Tube Length" as already mentioned, of 89 mm. The eye relief is 20 mm without Conrady stop which reduces it to about 13 mm.

The aberrational characteristics place the designed eyepiece, by Conrady's criteria, between good and fair and nearer to the former. This seems satisfactory in conjunction with the true field of over 20° and focal ratio of less than 2. The lateral colour becomes apparent only in landscape viewing in strong sunshine.

It is worth mentioning that, with this modified layout of the Ramsden eyepiece and as shown by the ray tracing, it is possible to achromatize the eyepiece without any obvious increase in other aberrations. To achieve this the field lens must be **stronger** (by about 30 per cent) than the eye lens. It was not done, firstly because it means a complete reversal of Conrady's precepts while still following other parts of his theory, secondly because the lateral colour would re-appear on focusing, and thirdly because the eye relief would be below the desired minimum (about 12 mm).

VI. CONCLUSIONS

The instrument described above may be capable of wider applications and its description useful to readers whose interests lie outside the auroral field. It is the presence of the movable parallactic graticule within the collimator that makes it an **auroral** collimator. Without the graticule it is simply a collimator-theodolite combination which gives direct readings of the spherical co-ordinates for the points on a photographic plate. It is not restricted to the horizontal system of co-ordinates, either. The precision can be greatly increased with higher magnification of the theodolite and by an arrangement in which the lens, while functioning as collimator, has the same position with respect to the photographic plate as it had while functioning as photographic objective.

There were cases in the astronomical field where requirements of speed took precedence over precision. Kapteyn (1896) designed a special instrument for rapid measurement of star positions on photographic plates and more recently Van Biesbroeck (1953, 1955) used a theodolite for measurement of photographic positions of asteroids. If the collimating principle described in this paper is utilized for the above purpose, the astrograph with which the plates were exposed provides an already mounted collimator. The only modification required is to the plateholder which must admit light from behind, e.g. stray daylight, as no condenser would be required for measuring star images. The astrograph is pointed with its objective towards the theodolite, which can be mounted on its tripod, by means of 180° rotation around the declination axis from the position it occupied during the exposure, which virtually provides the required 180° rotation of the plate **together** with the object glass as described in Section II. The fine alignment can be made rather with the levelling screws of the theodolite than with the slow motions of the astrograph. The field distortion, however small it is, and its irregularities depending on direction are corrected, and the differential refraction is the only remaining error in the apparent place.

VII. ACKNOWLEDGEMENT

Discussions of design with Dr. F. Jacka, Assistant Director (Scientific) of the Antarctic Division, are acknowledged and thanks are expressed to Dr. R. H. Stoy, H.M. Astronomer at the Cape, who kindly provided references quoted in the Conclusions.

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AURORAL PARALLACTIC PHOTOGRAPHY

PART 2

REDUCTION OF PARALLACTIC PHOTOGRAPHS

By

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(Manuscript received 14 April 1964)

ABSTRACT

Formulae are given for reduction of auroral photographs taken simultaneously from two bases. The error of the approximate formulae, introduced to save computing time, is strictly defined. The difference between the values of a spherical angle and its projection on the photographic plate is found too great to be ignored and a formula for the projected angle is given.

CONTENTS

I. INTRODUCTION

Parallactic photography of auroras has been undertaken by the Australian National Antarctic Research Expeditions at Macquarie Island. Pairs of photographs were simultaneously exposed at two bases about 30 km apart. Phototheodolites, described by Jacka and Ballantyne (1955), were used for the purpose. They consisted of cameras provided with vertical and horizontal circles indicating the co-ordinates (zenith distance and azimuth) of the centre of the graticule which was located at the focal plane of the camera objectives. The impression of the graticule appeared on the exposed photographic plates.

The task fell upon the writer to develop a method for measuring and reducing these photographs. Störmer's method, described by Störmer (1955) and Harang (1951), could not be used because it does not allow a precision of much better than 6 minutes of arc. The comparatively short base line of 30 km and the remoteness of the auroras, mostly at a few hundred kilometres distance, required a precision of measuring the points on a photographic plate of at least 1 minute of arc which would not be vitiated by the subsequent process of reduction. Such precision precludes the use of nomograms in most cases and makes computing inevitable. To speed up the process of computing, the ordinary formulae of spherical trigonometry were simplified within the limits of the required accuracy. To speed up the process of measuring the plates, a special instrument was designed which has been described in Part 1.

II. THE GENERAL PROBLEM

Figure 1 shows the earth, assumed spherical, with its centre at O and the two bases B_1 and B_2 separated by a distance corresponding to arc β.

A pointing from base B_1 on an arbitrary point P at the lower auroral border is represented by ray I_1 extending to infinity. I_2 and I_0 are infinite rays from B_2 and O parallel to ray I_1. The three parallel rays, representing an identical pointing in space, the first pointing, intersect at one point at infinity i.e. on the celestial sphere. The orthogonal projection of a point at infinity onto the spherical earth is directed towards its centre and is therefore I' on the surface of the earth.

In the horizontal system of co-ordinates at B_1 the pointing I_1 is defined through the zenith distance z_1 in the direction a_1 (measured here from the arc β). The great circle arc z_1 on the celestial sphere, connecting the zenith point of B_1 (OB_1 produced to infinity) with point I_1 (which is the same as point I_0 at infinity), projects onto the spherical earth as a great circle arc $B_1\,I' = z_1$. In the same way we obtain arc $B_2\,I' = z_2$ in the horizontal system of co-ordinates at B_2. It might help to add that the arcs

z_1 and z_2 are normal (orthogonal) projections on earth of the infinite rays I_1 and I_2. The infinite ray I_o projects on the earth as a single point I'.

We have thus on the spherical earth a triangle B_1 B_2 I' which is a projection (a replica) of a spherical triangle on the celestial sphere with the points: zenith of B_1, zenith of B_2 and pointing I_1.

The mode of representation by means of projecting the relations on the celestial sphere onto the spherical earth allows us to consider on the same figure the **pointings** on the aurora (as points at infinity) and the

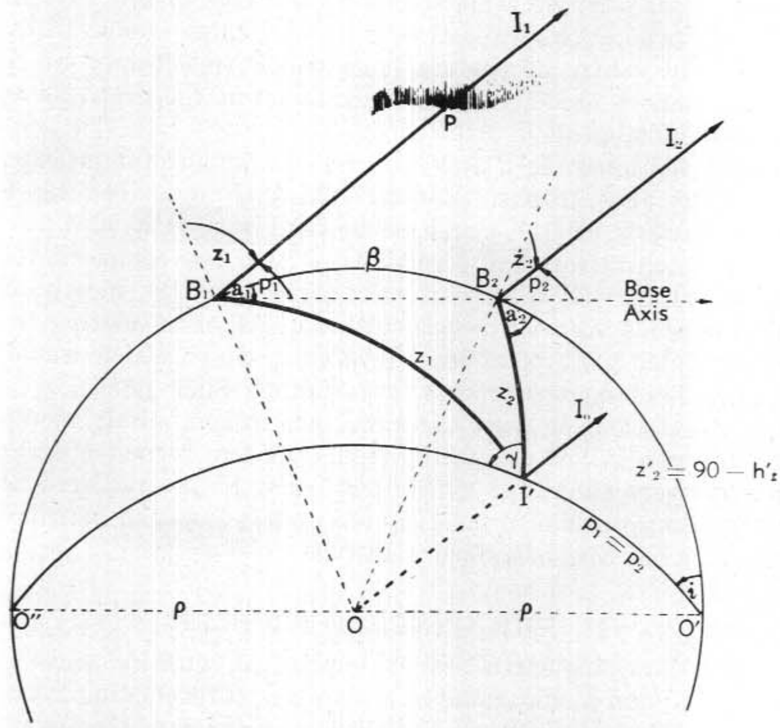

FIG. 1. Elements of Identical Pointing in Space.

auroral points themselves (points at finite distance) without faulting any relations. The spherical triangles necessary for the solution are shown together with the pointing rays, bases, and aurora. There is an element of duality in such a form. Ignoring the rays, we can consider the figure as representing the celestial sphere with the earth itself shrinking to a single point O. The latter contains then all points at finite distance: the aurora and the bases which are only distinguishable by their respective zenith points B_1 and B_2 on the celestial sphere. This alternative will be used later.

The spherical triangle B_1 B_2 I' gives us a transformation of co-ordinates of a point at infinity from the system of B_1 to the system of B_2, or (z_2,a$_2$) co-ordinates defining the pointing I_2 which is parallel to I_1. A direct pointing from B_2 on the auroral point P, arbitrarily selected from B_1, is not immediately possible because auroras have, in general, no identifiable discrete points, neither can point P be identified from B_2 by the intersection of the infinite ray I_1 with the lower auroral border because ray I_1 is not actually, physically, existent and is thus not visible from B_2. The solution of auroral problem, essentially Störmer's (loc. cit.), is based on rendering the pointing ray I_1 virtually visible from B_2.

The parallel rays I_1 and I_2 lie in an infinite plane containing the auroral point P and the Base Axis (chord B_1 B_2 on Figure 1). We consider another plane parallel to the first but containing ray I_o and the diameter of the earth O'' O' which is parallel to the Base Axis. The two parallel planes intersect with the celestial sphere (at infinity) along one and the same great circle arc which, projected back on earth, is the arc O' I' O''.

In order to make the pointing from B_2 on the auroral point P (as yet unknown), ray I_2 must be swung in the plane I_1 B_1 B_2 I_2 until it meets the lower auroral border. For this it is better to consider the relations on the sky and use the alternative way, the duality of representation. Both bases and the whole earth are now at a single point O, the sphere shown on the figure represents the sky (at infinity). Anything at finite distance is at point O, including the auroral point P. OI' is the pointing from B_1 on the auroral point; it is also the parallel pointing from B_2. We know the arc O' I' O'' along which we must swing the ray OI', referred to base B_2, in order to get the auroral point P selected at base B_1. The direction of this arc is more tangibly defined by the angle (γ) it forms with the vertical circle (base B_2) passing through point I'. The portion O'' I' of the arc O'' I' O' is the infinite ray from base B_1 as seen from base B_2, both bases being at point O on the figure. One can see the truth of this statement in two ways. Firstly, one can imagine an infinitesimal parallel shift of OI' to the left from O and view its projection on the sky from base B_2 at O. Secondly, one can revert to the first mode of representation with bases B_1 and B_2 on the earth with finite radius and points at infinity being out of reach. The ray I_1 is "seen" from base B_2 in the plane I_1 B_1 B_2 I_2 (edgewise) through the whole angle B_1 B_2 I_2 because the foot of the ray I_1 is in the direction B_2 B_1 and its tip joins the tip of the ray I_2 at infinity. One can see that it is the same as the arc O'' I' on the sky.

The ray I_1 is thus made visible from base B_2 because it can be actually drawn, or shown by any means whatever, on the auroral photograph taken at B_2 as a line passing through the known point I' at the known angle γ which it makes with the vertical circle. The intersection of this line with the lower auroral border re-identifies from base B_2 the auroral point selected at the base B_1.

It is to the credit of Störmer (1955) that he pointed out the usefulness of the concept of the direction of parallactic displacement for re-identification of auroral points.

The pointing from base B_2 on the auroral point P is shown on Figure 2 as an infinite ray II_2, the second pointing. It projects onto the earth normally as a great circle arc $B_2\ II' = z$. An infinite ray II_0, from the centre of the earth and parallel to II_2, is added to show more clearly that the arc z actually stops at the arc $O''\ I'\ O'$ and also to conform to the method of presentation as used at the beginning of this section for the first pointing. z is, of course, the zenith distance of the pointing II_2 and its horizontal direction, measured from arc β produced, is shown as a. P' is the normal projection on the earth of the auroral point P. It is located at the intersection of arcs z_1 and z because these arcs are the normal projections of the infinite rays I_1 and II_2 which intersect at the point P.

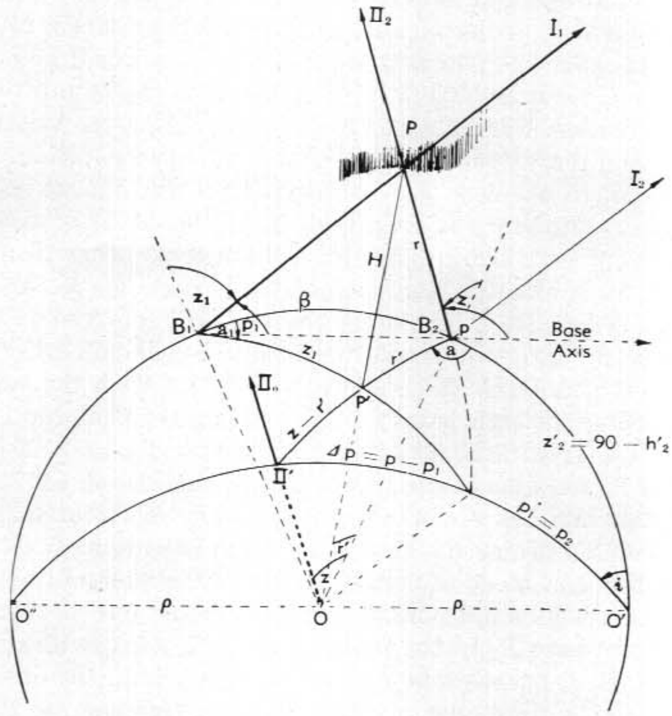

FIG. 2. Solution of Auroral Problem.

The complete solution is now obtained in principle. The distance, r, from B_2 to the auroral point P, can be calculated from the plane triangle $B_1\ B_2\ P$ which is shown on Figure 2. The distance to the projected auroral point P' shown as arc $r' = B_2\ P'$, can be calculated from the plane triangle $O\ P\ B_2$ and the auroral height $H = PP'$ is then given by the same triangle.

Before concluding this section another system of co-ordinates can be mentioned. It may be called the parallactic co-ordinate system as opposed to the horizontal co-ordinate system. In this system the rays representing pointings on the aurora lie in the plane containing the Base Axis (chord $B_1\ B_2$) and the auroral point. The direction of the ray is measured by the angle (p) it makes with the Base Axis and the slope of the plane (i) is measured with respect to the vertical plane containing the centre of the earth and the two bases. p-angles and their corresponding arcs on the sky projected back onto the earth and the i-angle are shown on both figures. Naturally $p_1 = p_2$ for an identical pointing in space. In general, p-arcs are the arcs of parallactic displacement and the angle (γ) they make with a certain vertical circle is the direction of parallactic displacement. $\Delta p = p - p_1$ is the amount of parallactic displacement or, in short, the parallax. The usefulness of i is increased by the fact that it is an invariant for all three stages of solution for each auroral point, the initial pointing from B_1, the parallel pointing from B_2 and the final pointing from B_2, because the parallactic displacement takes place in the plane defined by i.

III. NOTATION

Notation is explained in the appropriate sections in which new symbols occur, and as they are often confined to one section only there is no need for a general compilation. Instead, the principles adopted for notation are explained below.

As an aid to memory an arbitrary choice of letters was avoided. The established notation was followed, such as azimuth (a), zenith distance (z), height above the horizon $h = 90 - z$. Wherever possible the letter suggests the concept, in English, e.g. p for parallactic arc and i for inclination of the parallactic plane. Furthermore, a variant of the concept is denoted by a variant of the letter:

(a) true azimuth (reckoned eastwards from north),
(A) azimuth of the base line, a constant for each base,
(a) azimuth reckoned from the base line, instead of from the meridian, $\mathrm{a} = a - A$.

Primes are reserved for projected quantities, e.g. the projected values of spherical angles.

The pointing rays on Figure 2 were simply numbered, for the first and second pointing, as one and two in Roman numerals. Their subscripts 1, 2, 0, indicate from where the pointing was made: base B_1, base B_2, or the centre of spherical earth.

In the rest of the paper the subscript zero is largely reserved for the co-ordinates of the centre of the photographic plate (the optical axis) a_o, z_o, the values of which are known. Subscripts 1 and 2 refer to the bases, mainly for the constants such as A_1, A_2. The pointing from base B_1 is always denoted by a_1, z_1 because, in a way, it is a constant, the selected pointing. In the field of base B_2 the values a, z require in general no subscript and the same applies to the final pointing on the re-identified auroral point which was selected at base B_1. An exception is made for a_2, z_2, which result from the co-ordinates transformation of a_1, z_1.

IV. BASE CONSTANTS

An assumption of the spherical earth raises the question of the correct value of its radius and of the mode in which it is fitted to the geoid, assumed here to be an ellipsoid of rotation.

A horizontal plane is defined as a plane tangential to the surface of the earth at the location of the base whether the earth is assumed spherical or otherwise. The normal, perpendicular to the horizontal plane, points upwards to the zenith of the base, from which are measured the zenith distances on the sky, and downwards to the centre of the earth if the latter is assumed spherical, but not to the centre of the geoid. Hence, for the best coincidence of the two surfaces at the base, the radius of the spherical earth is equal to the radius of curvature of the geoid and the mode of the fit is eccentric. This concept is important whenever measurments on the sky are combined with those on the earth.

The sphere is, in fact, an osculating sphere fitted to the ellipsoid at a given point, the base, with the two surfaces having the smallest deviation from one another with distance (always less than 0·05 km within a distance of 1,000 km). Strictly speaking, there are two osculating spheres, one for each base, but this fact is quite immaterial by comparison with the variation of the radius of curvature of an ellipsoid with the direction. If we denote the radius of curvature in the plane of the meridian by ρ_ϕ (ϕ for geographical latitude) and the radius of curvature in the direction perpendicular to the meridian by ρ_λ (λ for longitude) than in any other direction, azimuth a, the radius of curvature can be calculated by Euler's formula of differential geometry:

$$\frac{1}{\rho_a} = \frac{\cos^2 a}{\rho_\phi} + \frac{\sin^2 a}{\rho_\lambda} \qquad (1)$$

There is thus a multitude of osculating spheres with their radii depending on azimuth. For auroral work it is generally sufficient to adopt a single value either as an average or as the value of ρ_a corresponding to the predominant direction in which the auroras are measured. Any variation of ρ in the process of auroral reduction implies, in a sense, a correction for the geoid.

A formula for the meridional radius of curvature, ρ_ϕ, is obtained by substituting elliptic relations in the well known curvature formula. The derivation of ρ_λ is simple by Meusnier's law and less so by equating the expressions for the ellipsoid and for the plane in the required orientation. Meusnier's law states that the radius of curvature for a skew section is equal to the projection of the normal radius of curvature on the skew plane. The skew plane, producing the skew section, contains only the tangent at the point on the ellipsoid while the normal plane contains the normal as well. A section of the ellipsoid with a skew plane parallel to the equator is a circle of radius x, where x is the cartesian co-ordinate of the point on the ellipsoid. The angle between the skew and the normal planes in this case is equal to the geographical latitude ϕ. Hence, by Meusnier's law $x = \rho_\lambda \cos\phi$ and by elliptic relations $x(1-e^2) = n\cos\phi$, where n is the length of the normal between the surface and the equatorial plane and e the eccentricity of the ellipsoid. We obtain ρ_λ by equating x in the two relations above.

It was found convenient, for computing purposes, to express both principal radii of curvature ρ_ϕ and ρ_λ through the length of the normal n and the equatorial radius of earth R_o:

$$n = \frac{R_o\,(1-e^2)}{(1-e^2\sin^2\phi)^{\frac{1}{2}}} \tag{2}$$

$$\rho_\phi = \frac{n}{1-e^2\sin^2\phi} \tag{3}$$

$$\rho_\lambda = \frac{n}{1-e^2} \tag{4}$$

The difference $(\rho_\lambda - \rho_\phi)$ is zero at the pole and reaches nearly 43 km at the equator.

A substitution of ρ by the more familiar radius vector R of the geoid is wrong because its values follow a different law and do not nearly represent earth curvature except within the range of: $20^\circ < \phi < 40^\circ$. For $\phi > 46^\circ$, $R < \rho_\phi < \rho_\lambda$.

Having adopted a constant value of ρ the arc β is determined from $\beta = b/\rho$, where b is the length of the base line measured by surveying. Also by surveying, the azimuth of the base line for each base, A_1 and A_2, must be known. Capital letters are chosen to denote these azimuths to distinguish between the constant and the variable values of the azimuth a. We have now four base constants: ρ, β, A_1, A_2 out of the total six.

The fifth base constant is shown on Figures 1 and 2 as z', the zenith distance of the base axis point projected on the celestial sphere. The Base Line is the length of the geodesic between the two bases measured along sea level, the Base Axis is a straight line connecting the two bases allowing for their actual locations above sea level. It is obvious from either figure, 1 or 2, that if the two bases are at the same height above sea level $z'_2 = 90 - \frac{1}{2}\beta$ and $z'_1 = z'_2 + \beta = 90 + \frac{1}{2}\beta$. Any difference between the heights of the two bases affects the value of z'.

Figure 3 shows the bases, B_1 and B_2, with the positive height difference ΔH. We can obtain the value of z'_2 from a simple relation: $(\rho + \Delta H)\sin(z'_2 + \beta) = \rho\sin z'_2$ after a few transformations. Firstly, z'_2 is nearly 90° and it is more convenient to transform it to the height above the horizon $h'_2 = 90 - z'_2$. Secondly, the two functions of h'_2, $\cos h'_2$ and $\sin h'_2$ are transformed to a single function, $\tan h'_2$, if both sides of the equation are divided by $\cos h'_2$. The resulting expression is:

$$\left(1 + \frac{\Delta H}{\rho}\right) = \frac{1}{\cos\beta + \tan h'_2 \sin\beta},$$

from which we have:

$$\tan h'_2 = \frac{1 - \cos\beta\left(1 + \frac{\Delta H}{\rho}\right)}{\sin\beta\left(1 + \frac{\Delta H}{\rho}\right)} = \tan\frac{\beta}{2} - \frac{\Delta H/\rho}{\sin\beta} \tag{5}$$

The second expression in (5) is derived from the first after multiplication of the numerator and denominator by $(1 - \Delta H/\rho)$ and ignoring $(\Delta H^2/\rho^2) < 10^{-6}$. An approximation sign is not used for that expression

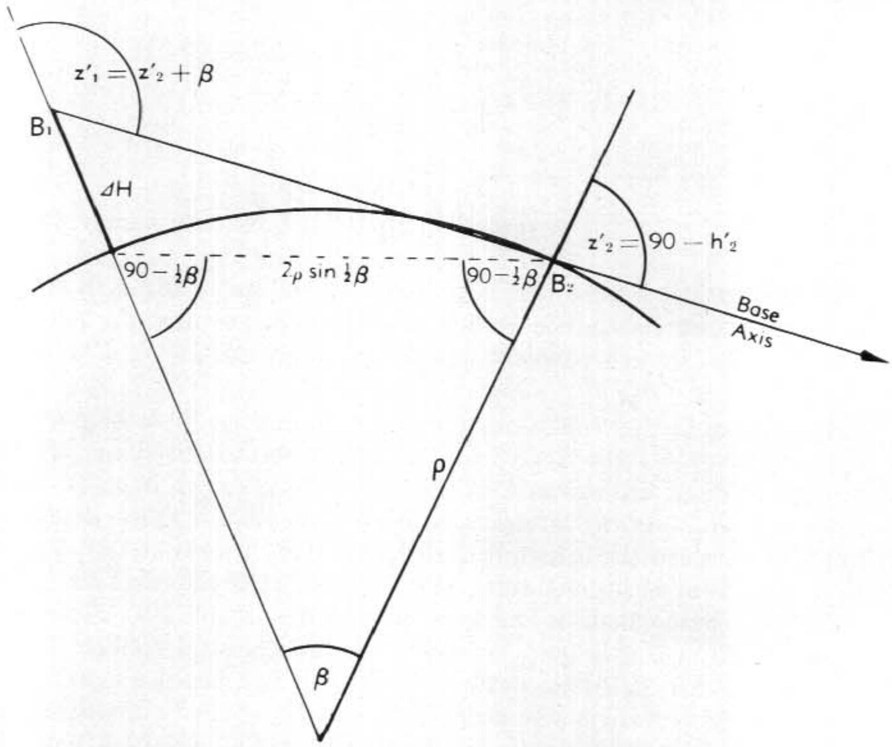

FIG. 3. Base Height Difference.

because its error is $(\Delta H^2/\rho^2) \cot\beta$ only. We obtain from (5) convenient approximation formulae

$$h'_2 \simeq \frac{1}{2}\frac{b}{\rho} - \frac{\Delta H}{b}$$

and

$$h'_1 = h'_2 - \frac{b}{\rho} \simeq -\left(\frac{1}{2}\frac{b}{\rho} + \frac{\Delta H}{b}\right) \tag{6}$$

which are actually quite accurate for most cases; the way they were arrived at obscures the fact that the approximation is equivalent to an assumption of $\cos h' = 1$ and of the equality of the length of the Base Axis $B_1 B_2$ and the Base Line $b = \rho\beta$. Whether or not the first assumption is tenable is immediately shown by the computed value of h' itself and the second assumption can be examined from a comparison of the length of the chord, $2\rho \sin \frac{1}{2}\beta$, shown by a broken line on Figure 3, with the length of the arc, i.e. the base line $b = \rho\beta$. The ratio:

$$\frac{\text{chord}}{\text{arc}} = \frac{\sin \frac{1}{2}\beta}{\frac{1}{2}\beta} \text{ is less than unity by}$$

$$10^{-6} \text{ for } \beta = 0{\cdot}030,\ b \simeq 190 \text{ km.}$$

$$10^{-5} \text{ for } \beta = 0{\cdot}078,\ b \simeq 500 \text{ km.}$$

In terms of length, the difference arc minus chord reaches roughly

1 cm for $b = 20$ km,
1 m for $b = 100$ km,
and 1 km for $b = 1000$ km.

The length of the Base Axis approximates to the length of the Base Line even better than the length of the chord does.

The derivative of (6) : $\delta h' = \left(\frac{1}{2}\frac{b}{\rho} + \frac{\Delta H}{b}\right)\frac{\delta b}{b} - \frac{\delta \Delta H}{b}$, showing the dependence of the accuracy of h' on the accuracy of the measured values of the base line b and the height difference ΔH, reveals an unexpected sensitiveness of h' to ΔH. For a 30 km base line, an error of 10 m in ΔH produces an error of one minute of arc in the value of h'.

There are two more points on which formula (6) provides immediate information. Firstly, for $\Delta H = 0$ (both bases at the same height above sea level) $h' = \frac{1}{2}\beta$ which was already known from Figures 1 and 2 and which is contrary to Harang's (1951) statement that $h' = 0$ for $\Delta H = 0$. Secondly, it gives the necessary value of ΔH that would produce $h' = 0$ and thus greatly simplify some of the reduction formulae:

$$\frac{\Delta H}{b} = \frac{1}{2}\frac{b}{\rho}, \text{ for } h' = 0.$$

The ΔH values are surprisingly high, 76 m (250 ft) for 31 km base line, increasing with the square of the base line to 784 m (2570 ft) for 100 km base line.

As representative values the constants for Macquarie Island are given below. The approximate formulae developed in subsequent sections were based, to a certain extent, on the small magnitudes of h' and β.

The subscripts (1) and (2) refer to the two bases on Macquarie Island, viz. Hurd Point and Buckles Bay respectively. Hayford's values $R_o = 6378{\cdot}4$ km and $e = 0{\cdot}082$ were used for computing ρ_ϕ and ρ_λ for the main base B_2.

$\phi_2 = -54°30'.0$ $\quad \rho_\phi = 6378{\cdot}1$ km $\quad \rho_\lambda = 6392{\cdot}6$ km

$A_1 = +13°2'.9$ $\quad A_2 = +12°57'.6$ $\quad b = 31{\cdot}201$ km

$\beta = 0{\cdot}00489 = 16'.8$ $\quad \Delta H = +7$ m in the sense $B_1 - B_2$.

$h'_1 = -0{\cdot}00267 = -9'.2$, for base B_1

$h'_2 = +0{\cdot}00222 = +7'.6$, for base B_2

For the azimuth $a = 13^\circ$, along the base line, the accurate value of ρ_a is 6379 km, but the value 6380 km was used instead for computing the β-value. The same value, $\rho = 6380$ km, was mostly used for auroral reduction though a small critical table, as shown below, was compiled in case higher precision appears desirable.

a	ρ_a	a
0°		−13°
	6380 km	
35		22
	6385	
55		42
	6390	
90		78

(a) stands for the true azimuth used in computing the table, and (a) for the azimuth measured from the base line, used in auroral reductions. For other values of (a) the values of ρ can be easily inferred from the table.

V. TRANSFORMATION OF CO-ORDINATES

Auroral points are selected at base B_1. The main base B_2 is used for the final solution, based on the re-identification of the selected points, as described in Section II. As the first step we need a transformation of co-ordinates for identical points at infinity, from the B_1-system of horizontal co-ordinates a_1, z_1, to the corresponding B_2-system a_2, z_2.

From triangle $B_1\,B_2\,I'$, Figure 1, we have by cosine formula:

$$\cos z_2 = \cos\beta\cos z_1 + \sin\beta\sin z_1\cos \mathrm{a}_1.$$

It is desirable to express the transformation of the z co-ordinate as a small correction δz to the known value of z_1, thus $z_2 = z_1 + \delta z$. The maximum value of δz cannot exceed the value of β, $\delta z \leqslant \beta$, and if the approximation of $\cos\beta = 1$ is permissible, then $\cos\delta z$ can be taken as unity also. Hence: $\cos z_1 + \beta\sin z_1\cos \mathrm{a}_1 = \cos\,(z_1 + \delta z) = \cos z_1 - \delta z\sin z_1$ and

$$\delta z \doteq -\beta\cos \mathrm{a}_1 \tag{7}$$

For $\cos \mathrm{a}_1 = \pm 1$, in the vertical plane of the base line, formula (7) is exact. For other values of a_1 it is approximate, with the maximum error of approximation corresponding to $\cos \mathrm{a}_1 = 0$. The errors are insignificant for $z_1 > 20^\circ$, being less than $0'{\cdot}1$, but they grow towards the zenith as shown in the small table below in which the maximum error of δz (i.e. for $\cos \mathrm{a}_1 = 0$), $\epsilon(\delta z)$, in minutes of arc, was computed in the sense of approximation minus exact value:

z_1	max. $\epsilon(\delta z)$
20°	−0′·1
10°	0·2
5°	0·4
2°	1·2
1°	2·2

With the precision requirements of better than 1 minute of arc, formula (7) can be used without restraint for $z_1 > 3°$. As can be seen from Figure 1, for $z_1 = 0$ (a_1 indeterminate) $z_2 = \beta = 16'8$, which is also the maximum value of δz and its constant value for $\cos a_1 = \pm 1$, irrespective of z_1.

To transform a_1 to the corresponding a_2-value we take the basic formulae of spherical trigonometry

$$\sin z_2 \cos a_2 = \sin z_1 \cos \beta \cos a_1 - \cos z_1 \sin \beta$$

and

$$\sin z_2 \sin a_2 = \sin z_1 \sin a_1$$

Dividing the first by the second we obtain

$$\cot a_2 = \cos \beta \cot a_1 - \frac{\sin \beta \cot z_1}{\sin a_1} \doteq \cot a_1 - \frac{\beta \cot z_1}{\sin a_1},$$

the last expression resulting from approximating $\cos \beta = 1$, $\sin \beta = \beta$.

We can now derive the required correction $\delta a = a_2 - a_1$ from the difference of cotangents which is shown in an abridged way below:

$$\frac{\beta \cot z_1}{\sin a_1} \doteq \cot a_1 - \cot a_2 = \frac{1 + \cot a_1 \cot (a_1 + \delta a)}{\cot \delta a}$$

$$= \frac{\tan \delta a}{\sin^2 a_1 + \sin a_1 \cos a_1 \tan \delta a}$$

$$\tan \delta a \doteq \frac{\beta \sin a_1 \cot z_1}{1 - \beta \cos a_1 \cot z_1} \tag{8}$$

For practical purposes formula (8) can be considered as exact. Its maximum approximation error occurs for $a_1 = 45°$ and even there it reaches the value of only $0'{\cdot}1$ for $z_1 = 1°$ and $1'{\cdot}2$ for $z_1 = 0°{\cdot}1$. The great circle arc equivalent of both these values, $\epsilon\,(\delta a) \sin z_1$, corresponds to an error of $0'.002$.

To speed up the reduction still more, formula (8) can be restricted to its numerator only:

$$\tan \delta a \doteq \beta \sin a_1 \cot z_1 \tag{9}$$

A comparison with formula (8) reveals the maximum approximation error of formula (9) again at $a_1 \doteq 45°$; they are tabulated below in the form of the great circle arc equivalent:

z_1	$\epsilon\,(\delta a) \sin z_1$
10°	0'·2
3°	0·8
1°	2·0

The above values show that formula (9) can be used without restraint for $z_1 > 3°$, which is the same limit as was found for formula (7).

What is achieved by these rather extreme simplifications is speed. Formula (9) can be represented by a convenient circular nomogram within which a small critical table, giving the values of formula (7), can be incorporated.

As will be subsequently shown, an alternative method of auroral reduction, using the system of parallactic co-ordinates described at the end of Section II, dispenses with the transformation of co-ordinates. However, the method requiring such transformation has its merits provided the latter is made as simple and fast as possible.

VI. PARALLACTIC CO-ORDINATES

As was mentioned at the end of Section II, any pointing on the sky can be defined through an angle p between the pointing ray and the base axis and through the inclination i of the plane containing the pointing ray and the base axis measured with respect to the vertical plane containing the base axis and the centre of spherical earth.

On Figure 1, taken to represent the celestial sphere, the p-angle becomes a p-arc and the inclination i becomes a spherical angle at the base axis point, i.e. the base axis produced to infinity.

The p-angles are still well defined if Figure 1 is viewed as representing the finite earth with the rays pointing to the sky. The angle between the two planes in space is less convenient to show but the spherical angle i, now considered to be projected from the sky on the earth, defines clearly that i is measured in the plane perpendicular to the base axis.

The sign convention is fixed to make the signs of $\sin i$ and $\sin \mathrm{a}$ the same, and p-arcs are always positive and reckoned from 0° to 180° from the point 0′, i.e. from the base axis-point.

Angle i is an invariant for all stages of solution of any one auroral point: (1) selected pointing from Base B_1, (2) identical pointing from Base B_2, (3) final pointing from Base B_2 on the re-identified auroral point, and is therefore useful though not indispensable.

The p-values are necessary in the final solution for computing auroral distance.

Figure 1 shows that p_1 is equal to p_2 whether they are viewed as represented by the same arc or as the angles which the two parallel rays make with the base axis. Figure 2 shows p for final pointing from base B_2, and the amount of parallactic displacement $\Delta p = p - p_1$.

From either of the two triangles $O'\ I'\ B_2$ or $O'\ I'\ B_1$ on Figure 1 we have

$\cos p = \sin h' \cos z + \cos h' \sin z \cos \mathrm{a} = h' \cos z + \sin z \cos \mathrm{a}$ in which the last expression, resulting from a substitution $\sin h' = h'$ and $\cos h' = 1$, is still exact to 10^{-5}, with the Macquarie values of h', and the approximation sign is redundant. The relation $z' = 90 - h'$ was taken into account in writing down the formula and the 1 or 2 indices are not used because p_1 is equal to p_2 and can be computed from the data of either base. The value of h' must correspond to the base whose values of a and z are used for the formula, i.e. for z_1 and a_1 the base B_1 value of h' must be used while for z_2 and a_2 the h' value must be that of base B_2.

A form of logarithmic transformation can be applied to the $\cos p$ expression by substitution of

$$m \sin M = h' \qquad m \cos M = \cos a \qquad \tan M = h' \sec a$$
$$m^2 = \cos^2 a + h'^2 = \cos^2 a \, (1 + h'^2 \sec^2 a) = \cos^2 a \sec^2 M$$

from which we have

$$\cos p = m \sin (z + M) = \sec M \cos a \sin (z + M).$$

The above expression offers no advantage in computing by comparison with the original formula unless we drop sec M, leaving

$$\cos p \simeq \cos a \sin (z + M) \qquad (10)$$

Although the above approximation represents only a slight gain in computing speed, its justification will become evident in the next section.

The error of the above approximation can be investigated by subtracting the exact formula from the approximate one, thus following the correct sense, by definition, of an error. We have then

$$(1 - \sec M) \cos a \sin (z + M).$$

The left hand side will be the difference of the two cosines of p which can be written as $\Delta \cos p = -\epsilon_p \sin p$ ignoring the higher terms of Taylor's expansion on the assumption that the error of p, ϵ_p, is sufficiently small. Hence

$$\epsilon_p = \frac{(\sec M - 1) \cos a \sin (z + M)}{\sin p}$$
$$= \frac{\sin (z + M) \, [(\cos^2 a + h'^2)^{\frac{1}{2}} - \cos a]}{[1 - (\cos^2 a + h'^2) \sin^2 (z + M)]^{\frac{1}{2}}} \qquad (11)$$

For $\cos a = 1$, along the base line, formula (10) is exact although we get from (11) $\epsilon_p = \frac{1}{2} h'^2 \tan (z + h')$. With the higher of the two h'-values, $h'_1 = 9'{\cdot}2$, $\frac{1}{2} h'^2$ expressed in minutes of arc is $0'{\cdot}01$, thus well below the accuracy of formula (11) which was developed from a logarithmic transformation of a formula based on the assumption of $\cos h' = 1$ (equivalent to $\frac{1}{2} h'^2 = 0$).

For $\cos a = 0$, $M = 90°$ and the error, $\epsilon_p = h' \cos z$, is at a maximum which places restriction on the use of the approximate formula (10). When a deviates from the base line by not more than 83° the error is kept well under $0'{\cdot}1$. Thereafter it grows rapidly reaching the value of $1'{\cdot}0$ at $a = 89° \; 16'$ or more explicitly: when the a-value is just within 1° of $\cos a = 0$. With the precision requirement of better than one minute of arc, formula (10) must not be used within the region of $a = 90° \pm 45'$ and $a = 270° \pm 45'$. As will be explained in the next section, the above restriction is not significant and the mode of the use of the formula can lead to no confusion.

To derive the formula for computing the value of i, the inclination of the plane containing the base axis and the selected auroral point, the two basic relations below can be used:

$$\sin p \cos i = \cos h' \cos z - \sin h' \sin z \cos a$$

and

$$\sin p \sin i = \sin z \sin a$$

These formulae refer to the same triangles on Figure 1 as were used for derivation of the cos p formula, and we can again assume cos $h' = 1$ and sin $h' = h'$. Dividing the first formula by the second we can write the final result as

$$\sin \mathrm{a} \cot i = \cot z - h' \cos \mathrm{a} \qquad (12)$$

Formula (12) can be used either with a_1 and z_1 values and the value of h'_1 or with a_2 and z_2 values, after the co-ordinates transformation, and the value h'_2. The subscripts 1 and 2 are therefore omitted from the formula. With the Macquarie values of h' formula (12) is exact to 10^{-5} and no simplifying approximation is applied because it is in fact a dual purpose formula. Application of a transformation would result in two formulae and cancel a certain advantage of using one and the same formula for two purposes unless the approximation consisted in the omission of the h' cos a term. The latter would be permissible with very small values of h' because the maximum error of such approximation lies at $\mathrm{a} = 45°$, or more explicitly at cos $\mathrm{a} = 0{\cdot}7$ (a much used region), where the error would reach the value of h' on the horizon and would still be $\frac{1}{3} h'$ for $z = 45°$.

If cot i is computed with a_1, z_1 and h'_1 values we can, by judicious selection of an a-value referring to base B_2, compute a corresponding z-value from the same formula (12) and the same cot i value and without a tabular determination of the value of i itself, but using of course the value of h'_2. Herein lies the duality of use of formula (12). We obtain then a point in the field of the photographic plate, expressed through the pair of values a and z which is not the point a_2, z_2 (identical pointing in space or simply the co-ordinates transformation), but a point which still lies on the correct p-arc as defined by i and that is all that is actually wanted. The need for a co-ordinates transformation is thus by-passed, though in some cases this might lead to a trial and error method as some values of a might result in z-values which are not within the field of the photographic plate at all.

Moreover, if two or three points for base B_2 are computed by formula (12), then a line drawn through these points represents the p-arc itself which in effect is the ray from base B_1 pointing on the selected auroral point, as that ray is visible in the field of the B_2-base plate. As the intersection of the line, represented by the p-arc, with the lower auroral border re-identifies the point selected at base B_1, the use of formula (12), as described, is an auroral method in itself, though it is not actually used in this paper.

VII. THE DIRECTION OF PARALLACTIC DISPLACEMENT

The p-arcs shown on Figures 1 and 2 are the arcs along which a point at finite distance will be parallactically displaced when observed from the two bases B_1 and B_2. They will be referred to in this paper as the lines of parallactic displacement or p-lines.

The direction of parallactic displacement is defined by an angle, γ, between the line of parallactic displacement and the vertical circle ($B_2 I' = z_2$ on Figure 1) which intersect at a given point on the sky or on the photographic plate.

Angle gamma is measured clockwise from a point on the vertical circle lying above the p-line, as is shown on Figure 1 if the latter is considered to represent the celestial sphere with the observer at point O. The above sign convention, clockwise for $\gamma > 0$, applies to the sky, to the projection of an auroral photograph on the screen and to the positive contact print, but it must be reversed when the original photographic plate is viewed directly "emulsion up", or in the measuring instrument (Part 1), in which case the positive direction is counter-clockwise.

On the sky γ is a spherical angle whose value is computed from the spherical relations. On the flat photographic plate this angle becomes a projection of the spherical angle γ which will be denoted here as γ'. An assumption of $\gamma' = \gamma$ is possible only when the precision requirements are abnormally low or when auroral work can be restricted to the immediate proximity of the optical axis on the photographic plate.

A precision requirement of one minute of arc implies that the great circle arc equivalent of the error in the γ-angle, ϵ_γ, should not exceed one minute, i.e. $\epsilon_\gamma \sin \Delta p = 1'$, where Δp, the amount of parallactic displacement, was generally less than 6° on Macquarie plates. The above is the same, in concept, as the great circle arc equivalent of an azimuth error, $\epsilon_a \sin z$, where z is the zenith distance of the measured point. In the immediate proximity of the zenith the azimuth error is naturally very large but the effective error is its great circle arc equivalent. With $\sin \Delta p \simeq 0{\cdot}1$ we can set the limit $\epsilon_\gamma < 10'$, which would apply either to the error caused by an approximate formula or to the difference $\gamma - \gamma'$, as the case may be.

A formula for the spherical angle γ is needed first as a basis for further development to allow for γ'. Applying the same method as was used for the derivation of formula (12) in the preceding section we have from relations shown in triangle $I' O' B_2$ in Figure 1:

$$- \sin p \cos \gamma = \sin h' \sin z - \cos h' \cos z \cos \mathrm{a}$$

$$+ \sin p \sin \gamma = \cos h' \sin \mathrm{a}$$

$$\cot \gamma = \frac{\cos \mathrm{a} \cos z - h' \sin z}{\sin \mathrm{a}} \tag{13}$$

The substitution of h' for $\tan h'$ in the above formula is trivial because for base lines with large values of h' it can be considered a shorthand notation for a constant value of $\tan h'$, while for Macquarie Island $(\tan h' - h') < 10^{-8}$. Formula (13) is thus quite exact.

We apply the same transformation to formula (13) as was used for the derivation of formula (10) of the preceding section and show it below in an abridged way as:

$$\tan M = \frac{m \sin M}{m \cos M} = \frac{h'}{\cos \mathrm{a}} = h' \sec \mathrm{a};$$

$$m^2 = \cos^2 \mathrm{a} + h'^2 = \cos^2 \mathrm{a} \sec^2 M$$

$$\cot \gamma = \frac{m \cos (z + M)}{\sin \mathrm{a}} = \sec M \cot \mathrm{a} \cos (z + M)$$

$$\cot \gamma \doteq \cot \mathrm{a} \cos (z + M) \tag{14}$$

Formula (13) requires about 50 per cent more computing and machine operations than formula (14). The approximate formula (10), for cos p, showed only a slight computing gain over the exact cos p formula but this gain is now increased by the fact that it shares the same function M with formula (14). The values of M are convenient to tabulate in a critical table requiring no interpolation as long as the value of a is not too close to 90° or 270° and herein, incidentally, lies the elimination of a possible mistake of using the approximate formula (10) in the region where its accuracy is insufficient. However, in the direction at right angles to the base line a re-identification of the selected auroral point becomes anyway difficult or impossible, owing to the near grazing incidence of the line of parallactic displacement with respect to the auroral border without any distinct point of intersection.

The approximation error in formula (14) can be determined by the same differencing method as was applied to formula (10) in the preceding section. We have thus $\Delta \cot \gamma = - \epsilon_\gamma / \sin^2 \gamma = (1 - \sec M) \cot \mathrm{a} \cos (z + M)$. Expressing $\sin^2 \gamma$ and sec M in terms of a and z we obtain

$$\epsilon_\gamma = \frac{\cos (z + M) \{[(1 + h'^2) \cot^2 \mathrm{a} + h'^2]^{\frac{1}{2}} - \cot \mathrm{a}\}}{1 + \cos^2 (z + M) [(1 + h'^2) \cot^2 \mathrm{a} + h'^2]} \tag{15}$$

The denominator in (15) may be written as $1 + \cot^2 \mathrm{a} \cos^2 (z + M)$ without introducing any inaccuracy but the numerator cannot be similarly simplified because the formula would then indicate a zero error throughout.

The error of the approximate formula (14) is strictly zero only for $\tan \mathrm{a} = 0$ and also for $\cos (z + M) = 0$, but this is a less clearly defined region because M depends on a. The indefiniteness of ϵ_γ, when both tan a and $\cos (z + M)$ are zero, is due to the location of the point in question at the base axis point (on the sky) itself where angle γ cannot be defined, in the same way as the azimuth cannot be defined at the zenith point.

The maximum of ϵ_γ, inferred from formula (15) and confirmed by equating the derivative of ϵ_γ to zero, occurs at $\cot \mathrm{a} = 0$ where the values of M and γ are defined by $\cot \mathrm{a} = \cot M = \cot \gamma = 0$. Hence,

$$|\epsilon_\gamma|_{\max} = h' \sin z.$$

Only the base B_2 value of h' (7′·6) is applicable to γ and the maximum approximation error imposes no actual restriction on the use of formula (14). The restriction is imposed rather by auroral considerations of "grazing" incidence mentioned before when auroras are observed at nearly right angle to the base line and by the tabulation of M. The latter varies from 8′ to 59′ with the variation of a from 0° to 83° but thereafter, for the remaining 7° of a-variation, increases rapidly to $M = 90°$. A tabulation limit of $M \leqslant 60'$ corresponds to the range of a-values, in four quadrants, for which the gamma error is less than 0′·1, much more accurate than actually required.

To derive the difference between the spherical angle γ and its projected value γ' on the photographic plate we can assume, in the first approximation, a gnomonic projection. The latter implies a distortionless field with all great circle arcs, the geodesics, imaged as straight lines, and the centre of projection on the optical axis.

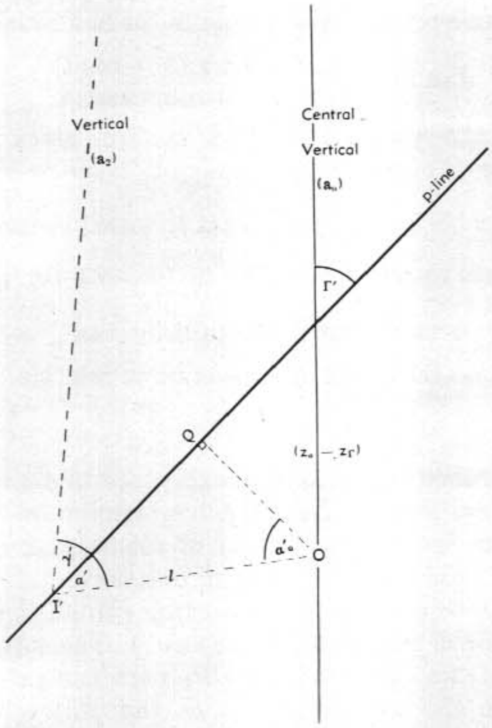

FIG. 4. Direction of Parallactic Displacement.

We shall first consider a right-angled spherical triangle $O\ Q\ \Gamma'$ on the sky with the vertex O on the optical axis of the camera lens. The gnomonic projection of this triangle is shown on Figure 4 with O on the optical axis point of the photographic plate which is the centre of projection. Owing to the basic property of gnomonic projection the orthogonality (with respect to central ray) at point Q and the value of α_o (angle between the central rays) are preserved. Projected angles on Figure 4 are primed and $\alpha'_o = \alpha_o$ but $\alpha' \neq \alpha$, where α_o and α are the values of the spherical angles. For a right angled spherical triangle we have, by Napier's formula,

$$\cot \alpha_o = \tan \alpha \cos l$$

and after projection, $\cot \alpha_o = \tan \alpha'$, because in the plane triangle $\alpha_o = \alpha'_o = 90 - \alpha'$. Hence, the simple relation between the spherical angle α and its gnomonic projection α', the angle projection formula, is

$$\tan \alpha' = \tan \alpha \cos l$$

where l is the distance (in degrees) of the vertex of α (point I') from the optical axis on the sky. Already the above relation shows sufficiently clearly that the difference between α' and α is not negligible. To obtain the difference of $(\alpha - \alpha')$ directly we subtract both sides of the equation from $\tan \alpha$ and write the result, in reversed order, as

$$\tan \alpha \, (1 - \cos l) = \tan \alpha - \tan \alpha' = \tan \, (\alpha - \alpha') \; (1 + \tan \alpha \tan \alpha').$$

Substituting the angle projection formula in the last bracket we have

$$\tan \, (\alpha - \alpha') = \frac{\tan \alpha \, (1 - \cos l)}{1 + \tan^2 \alpha \cos l}$$

The derivative of $\tan (\alpha - \alpha')$, equated to zero, gives its maximum for $\tan^2 \alpha = \sec l$. Hence

$$\tan \, (\alpha - \alpha')_{\max} = \tfrac{1}{2} \, (1 - \cos l) \sqrt{\sec l} \doteq \tan^2 \frac{l}{2}$$

The last approximate expression refers to $(\alpha - \alpha')$ for $\tan \alpha = 1$ which is nearly the same as a maximum. Substituting $\tan \frac{l}{2} \doteq \frac{l}{2}$ and converting into degrees we obtain the simplest approximate relation

$$0^\circ < | \, \alpha - \alpha' \, | < \frac{l^2}{229}$$

in which the α-difference and l are both expressed in degrees and the result is only $2 - 3$ per cent smaller than the true maximum value for $l < 25^\circ$. The maximum error (i.e. for $\tan \alpha \doteq 1$) of assumption of $\alpha' = \alpha$ reaching $\frac{1}{4}^\circ$ for $l = 7\frac{1}{2}^\circ$ and 1° for $l = 15^\circ$ is difficult to ignore.

We must now seek to apply the preceding results to the angle γ whose projection, γ', is shown on the same Figure 4. The latter represents the photographic field from the main base B_2 with central vertical, azimuth a_o, passing through the optical axis at O, the centre, with co-ordinates a_o, z_o. The point I' corresponds to a point with (a_2, z_2) co-ordinates which are obtained by co-ordinates transformation from a_1, z_1 referring to base B_1. The vertical from zenith to I' is indicated as vertical (a_2) and its length on the sky is z_2 degrees. The line of parallactic displacement, the p-line, is drawn across the field through point I'.

The full angle at I', on the sky, between the vertical circle and the l-arc representing the distance of I' from the optical axis can be written, for the particular configuration of Figure 4, as $(\gamma + \alpha)$. The value of $(\gamma + \alpha)$ can be computed from the spherical relations and its projection from $\tan \, (\gamma' + \alpha') = \tan \, (\gamma + \alpha) \cos l$. To derive the last expression from first principles one must draw a perpendicular from O to the Vertical (a_2).

The angle projection formula is applicable only to angles having one side passing through the centre, i.e. $(\gamma + \alpha)$ and α, but not γ. One can obtain γ' from $(\gamma' + \alpha') - \alpha'$, after the spherical value α was obtained from $(\gamma + \alpha) - \gamma$, but such procedure is not very useful for the solution because of the amount of computing it requires. It is, however, theoretically useful as it shows that $(\gamma - \gamma')$ can differ by up to 100 per cent from the values given by the preceding formulae for $(\alpha - \alpha')$, because γ' is *basically* a difference of two projected angles. To consider the effect of such a difference let us denote $(\alpha + \gamma)$ by K_1 and α by K_2. If $K_1 = 45° + x$ and $K_2 = 45° - x$ then $(K_1 - K'_1) \simeq (K_2 - K'_2)$ because K_1 and K_2 are symmetrically disposed around the maximum of $(K - K')$ at $K \simeq 45°$. On the other hand, for $K_1 = 90° + x$ and $K_2 = 90° - x$ we have $(K_1 - K'_1) = -(K_2 - K'_2)$ because K_1 and K_2 are symmetrically disposed around the minimum, $(K - K') = 0$, at $K = 90°$. Substituting $(\alpha + \gamma)$ for K_1 and α for K_2 and $\frac{1}{2}\gamma$ for x we obtain:

$$(\gamma - \gamma') \simeq 0 \text{ for } \alpha = 45° - \tfrac{1}{2}\gamma$$
$$(\gamma - \gamma') = -2\,(\alpha - \alpha') \text{ for } \alpha = 90° - \tfrac{1}{2}\gamma$$

while for $\alpha = 90°$ or $\alpha + \gamma = 90°$ the difference $|\gamma - \gamma'|$ will agree with the values given by the preceding formulae for $(\alpha - \alpha')$. The reader might find it useful to make a few sketches to verify the above statements. The maximum of $(\gamma - \gamma')$ goes now up to 1° for $l \simeq 11°$ and up to 2° for $l \simeq 15°$.

Angle γ need not be computed for the point I' resulting from the transformation of co-ordinates. In fact even the latter is not strictly necessary and its only convenience consists in obtaining a point (a_2, z_2) which definitely lies on the correct p-line. Angle γ may be computed for any other point lying on the line of parallactic displacement and we select its intersection with the central vertical and, to make a distinction, denote the angle by Γ. The co-ordinates of the point of intersection are a_o, z_Γ, and the angle projection formula is directly applicable to Γ with $l = z_o - z_\Gamma$.

With the known co-ordinates of the centre (a_o, z_o) we must now derive formulae for computing z_Γ, Γ and Γ'. If the inclination i of the parallactic plane has been computed, either with a_1, z_1 or their transformed values a_2, z_2, we can use formula (12) with the known cot i value and write it as:

$$\cot z_\Gamma = \cot i \sin a_o + h' \cos a_o \tag{16}$$

If cot i has not been computed and the ordinary routine of co-ordinates transformation is followed, the cot z_Γ formula can be obtained by equating the unknown, but constant, value of cot i:

$$\cot i = \frac{\cot z_2 - h' \cos a_2}{\sin a_2} = \frac{\cot z_\Gamma - h' \cos a_o}{\sin a_o}$$

from which we have

$$\cot z_\Gamma = \frac{\sin a_o \cot z_2 + h' \sin (a_2 - a_o)}{\sin a_2} \tag{17}$$

The h'-value in (16) and (17) must, of course, refer to base B_2.

Formula (14) in application to Γ can be written as

$$\tan \Gamma \simeq \frac{\tan a_o}{\cos (z_\Gamma + M_o)}$$

which, in combination with the angle projection formula

$$\tan \Gamma' = \tan \Gamma \cos (z_\Gamma - z_o)$$

gives directly a formula for $\tan \Gamma'$:

$$\tan \Gamma' \simeq \tan a_o \frac{\cos (z_\Gamma - z_o)}{\cos (z_\Gamma + M_o)} \qquad (18)$$

where M_o is taken from a critical table previously described or, by definition, $\tan M_o = h' \sec a_o$. The approximation error of formula (18) is the same as was derived for (14). For all auroral points selected at base B_1, $\tan \Gamma'$ is computed with the single variable z_Γ; a_o, z_o and M_o are constant for the photographic plate of base B_2.

The value of Γ and the values of γ referring to various points on the line of parallactic displacement differ from one another because they refer to different vertical circles. The same applies to Γ' and the various γ'values, i.e. they differ in their values because they refer to different projected vertical circles. If computed γ' values all referred to vertical lines parallel to the central vertical their values would be identical, constant along the p-line, and equal to Γ'. Hence Γ', computed specifically for the point (a_o, z_Γ), may be made to refer to any other point provided the latter is known to lie on the correct p-line. A line set at an angle equal to Γ' with respect to the central vertical and subjected to a parallel shift becomes the line of parallactic displacement when it passes through a point known to be on that line. The point would usually be a_2, z_2, unless we compute some other, using (12). This is important because the point a_o, z_Γ, need not be within the field of the photographic plate and often is not because the maximum value of $(z_\Gamma - z_o)$ is 90° when the p-line runs parallel to the central vertical. An assumption of gnomonic projection over such a wide range remains true as long as the camera optics does not interfere with it and the effect of field distortion need be considered only within the field of the photographic plate.

A study of an accurate lens calibration, of a Leitz Summarit lens, compared analytically with gnomonic projection, resulted in slight corrections, $\delta\Gamma'$, expressing the difference between the lens-projected angles and gnomonic angles

l:	10°	15°	20°
$\delta\Gamma'$:	±0°·0	±0°·1	±0°·2

The values shown above refer to $35° < |\Gamma'| < 55°$ only; outside these limits they drop sharply. The signs follow the sign of Γ', i.e. the acute angle is always *increased*. The values of l represent the distance from the optical axis of the point to which the p-line was fitted, not the (a_o, z_Γ)-point The total amount of correction is $k\delta\Gamma'$ where k is an asymmetry factor ranging from zero in a fully symmetrical case, when the point to which the p-line is fitted and the lower auroral border are more or less symmetrically disposed with respect to the centre O, to unity value when both the point and the intersection of the p-line with the auroral border are in the same quadrant off the centre. Factor k allows for the non-linearity of the actual line of parallactic displacement as imaged by the lens which, during the measure, is substituted by the straight line. As the corrections are so small with this particular lens their application is hardly necessary.

VIII. COMPUTATION OF AURORAL DISTANCE AND HEIGHT

The p-line set at an angle Γ' with respect to the central vertical, so as to pass through a point known to lie on the line of parallactic displacement, (a_o, z_Γ) or (a_2, z_2) or any other by formula (12), intersects the lower auroral border at the point which was selected from base B_1. The co-ordinates of the point of intersection (a, z) can now be read off. We have thus two pointings, (a_1, z_1) and (a, z), one from each base, on the same auroral point, and the final reduction, the computation of auroral distance and height, becomes a straightforward surveyor's problem.

Auroral distance, defined as the length of a ray r from base B_2 to the auroral point, can be determined from the plane triangle $B_1\ B_2\ P$, shown on Figure 2, by the relation $r \sin (p - p_1) = b \sin p_1$ in which a substitution of $b = \beta\rho$ for the length of the Base Axis is justified by the small difference between the two values as tabulated in Section IV. In the above relation p is the value (length) of the arc of parallactic displacement, computed by formula (10), for the final pointing from base B_2. p_1 is already known from the first pointing from base B_1 or its equivalent pointing (identical in space) from base B_2, $p_1 = p_2$. The difference $(p - p_1)$ is a measure of parallax.

Dividing the relation above by the radius of earth curvature, ρ, which is a constant for base B_2, computed by formulae (1) to (4) of Section IV, and expressing b/ρ as β we have

$$\frac{r}{\rho} = \frac{\beta \sin p_1}{\sin (p - p_1)} \tag{19}$$

The plane triangle $O\ P\ B_2$, on the same Figure 2, shows a relation $\rho \sin r' = r \sin (z - r')$ in which r' is the distance from B_2 to the projection of the auroral point on the earth defined as great circle arc $B_2\ P'$ or the angle r' shown at the centre of the earth. Developing $\sin (z - r')$, dividing both sides of the equation by $\cos r'$, and transforming, we obtain

$$\tan r' = \frac{\frac{r}{\rho} \sin z}{1 + \frac{r}{\rho} \cos z} \tag{20}$$

The same triangle gives two relations for auroral height above sea level: $(\rho + H)\sin(z - r') = \rho \sin z$ and $(\rho + H)^2 = \rho^2 + r^2 + 2\rho r \cos z$ from which we have

$$1 + \frac{H}{\rho} = \frac{\sin z}{\sin(z - r')} = \left(2\frac{r}{\rho}\cos z + \frac{r^2}{\rho^2} + 1\right)^{\frac{1}{2}} \tag{21}$$

Of the alternatives in formula (21) the second is more convenient as it requires less tabular work and r', in general, is only wanted for computing the geographical location of the projected auroral point, which is often superfluous. Furthermore, the extraction of the square root is particularly simple in this case because the under-root value is $1 + n$ with $n < 0{\cdot}04$. The square root is then equal to the mean value between $1 + \frac{1}{2}n$ and the result of division: $\dfrac{1+n}{1+\frac{1}{2}n}$.

Atmospheric refraction needs to be considered only in the values of z in formula (21), and (20) if the latter formula is used. In general, all visible points on the photographic plate are affected by refraction but, between them, the effective part is that of the differential refraction the magnitude of which is not large enough to make this refinement worth while. The value of H in formula (21) will, however, react slightly to the refraction effect in the final value of z. The refraction effects were all extensively tested in experimental computing.

The value of H, the auroral height, refers strictly not to the sea level but to the level of the main base B_2. If the height of B_2 is large enough to be considered, its value must be added to H to reduce the latter to the sea level.

The geographical co-ordinates of the projected auroral point P' can be computed from the polar spherical triangle (not shown) with the sides representing the colatitude of base B_2, the colatitude of P', and the distance r' between B_2 and P'. The angle at the pole, opposite r', is the difference between the longitudes of B_2 and P'. The angle at B_2 is the true azimuth (a) of P' as measured from B_2:

$$a = \mathrm{a} + A_2$$

where (a) refers to the final pointing from base B_2 on the aurora and A_2 is the base constant defined in Section 4.

The basic formulae of spherical trigonometry give the complete solution as:

$$\cos\phi \sin \Delta\lambda = \sin a \sin r' \tag{22}$$

$$\cos\phi \cos \Delta\lambda = \cos\phi_2 \cos r' - \sin\phi_2 \sin r' \cos a \tag{23}$$

$$\sin\phi = \sin\phi_2 \cos r' + \cos\phi_2 \sin r' \cos a \tag{24}$$

where the value of r' is known by formula (20), ϕ and ϕ_2 are the geographical latitudes (not colatitudes) of P' and B_2 respectively, reckoned negative in the southern hemisphere, and $\Delta\lambda = \lambda - \lambda_2$ with λ denoting the longitudes which are reckoned positive eastwards from Greenwich meridian.

Generally it is sufficient to use formulae (24) and (22) only, unless the base is very near the pole and there is a possibility of $\Delta\lambda$ exceeding 90°. In the latter case the first two formulae (22) and (23), computed separately, fix the quadrant of $\Delta\lambda$ without ambiguity because, with $\cos\phi$ always positive, they give the signs of the sine and of the cosine of $\Delta\lambda$. Finally, dividing the computed numerical values of (22) by those of (23) we get the values of $\tan\Delta\lambda$ with the already known quadrant of $\Delta\lambda$. The value of $\cos\phi$ can be then obtained by (22).

As an alternative we can divide the formulae, (22) by (23), and obtain

$$\tan\Delta\lambda = \frac{\sin a \tan r'}{\cos\phi_2 - \sin\phi_2 \cos a \tan r'}$$

and $\cos\phi$ by (22).

IX. THE ERROR OF DETERMINATION OF AURORAL HEIGHT

Taking the second expression in formula (21) the error of auroral height ϵ_H can be determined strictly as an error of a function according to the error theory

$$\epsilon_H^2 = \left(\frac{\partial H}{\partial r}\right)^2 \epsilon_r^2 + \left(\frac{\partial H}{\partial z}\right)^2 \epsilon_z^2$$

Let $K = \left(1 + \frac{H}{\rho}\right)$, $\epsilon_H = \rho\epsilon_K$, and formula (21) squared:

$$K^2 = 2\frac{r}{\rho}\cos z + \frac{r^2}{\rho^2} + 1$$

in which z refers to the final pointing from base B_2 and $\frac{r}{\rho}$ is defined by formula (19).

From the K^2 formula above we find expressions for

$$\cos z = \frac{(K^2-1) - \frac{r^2}{\rho^2}}{2\frac{r}{\rho}} \text{ and, from it, for } \frac{r}{\rho} + \cos z = \frac{(K^2-1) + \frac{r^2}{\rho^2}}{2\frac{r}{\rho}}$$

which we shall use for a substitution.

In order to obtain a less cumbersome formula we transfer the whole of the error to the final pointing from base B_2. This is permissible on two counts. Firstly the pointing from base B_1 is rather arbitrary and can be considered exact while the final pointing on the intersection of the p-line with a diffuse auroral border involves an error of identification of the selected point. Secondly, one can revert to the concept of the sum of the errors of the initial and the final pointings by merely doubling the error of the final pointing. The pointing error is anyway an estimate unless it is rigorously determined from a multiple measure of the same aurora. Hence, in formula (19), we consider p_1 to be exact and the parallax $\Delta p = p - p_1$ affected by an error due to the error of p. In the parallax error, $\epsilon_{\Delta p}$, p need not be computed because the measurement itself is made along the p-line. It is this error that one has to estimate.

Differentiating formula (19) and substituting ϵ_r and $\epsilon_{\Delta p}$ for the differentials we obtain:

$$\frac{\epsilon_r}{r} = -\epsilon_{\Delta p} . \cot \Delta p \approx -\frac{\epsilon_{\Delta p}}{\Delta p}$$

The above result, particularly in its approximate form, is significant as it shows that the percentage errors in distance and in parallax are roughly the same. $\epsilon_{\Delta p}$ is more or less constant, depending on the quality of the photographic plate alone, hence with a small parallax Δp no high accuracy can be achieved in determining r, which is rather obvious on simple geometrical considerations as well.

Differentiating K^2 partially by z and by $\frac{r}{\rho}$ we obtain

$$\frac{\partial K}{\partial z} = -\frac{\frac{r}{\rho}\sin z}{K} \text{ ; and } \frac{\partial K}{\partial \left(\frac{r}{\rho}\right)} = \frac{\frac{r}{\rho} + \cos z}{K} = \frac{(K^2 - 1) + \frac{r^2}{\rho^2}}{2K\frac{r}{\rho}}$$

Substituting the above relations into the error formula and taking into account that $\epsilon_K^2 = \frac{\epsilon_H^2}{\rho^2}$ and $\epsilon_{r/\rho}^2 = \frac{\epsilon_r^2}{\rho^2}$

we have:

$$\frac{\epsilon_H^2}{\rho^2} = \frac{\left[(K^2 - 1) + \frac{r^2}{\rho^2}\right]^2}{4\,K^2\,\frac{r^2}{\rho^2}} \cdot \frac{r^2}{\rho^2} \cdot \epsilon_{\Delta p}^2 . \cot^2 \Delta p + \frac{\frac{r^2}{\rho^2}\sin^2 z}{K^2} . \epsilon_z^2$$

The $\frac{r^2}{\rho^2}$ factors cancel out in the first term and we can take $4K^2$ and $\epsilon_{\Delta p}$ out of the whole expression writing the result as:

$$\epsilon_H = \pm \frac{\rho\epsilon_{\Delta p}}{2K}\left\{\left[(K^2 - 1) + \frac{r^2}{\rho^2}\right]^2 \cot^2 \Delta p + 4\frac{\epsilon_z^2}{\epsilon_{\Delta p}^2}\cdot\frac{r^2}{\rho^2}\sin^2 z\right\}^{\frac{1}{2}}$$

The error in the final pointing in zenith distance, ϵ_z, is smaller than the parallax error $\epsilon_{\Delta p}$ along the p-line which often runs across at a slope so that $4\frac{\epsilon_z^2}{\epsilon_{\Delta p}^2}$ might approach unity. However, whatever assumption we make for the second term is of little importance because the $\cot^2 \Delta p$ factor of the first term makes the latter more than by an order of magnitude greater than the second term, usually 30 or 100 times or more. The bracketed value $(K^2 - 1)$ in the first term may, in actual computing, be taken as a constant equal to 0·033. In the factor to the square root we shall consider $2K = 2{\cdot}033 =$ const., $\epsilon_{\Delta p} = 0{\cdot}000291$ (one minute of arc) and ρ about 6400km. The above values make the factor equal to 0·915 km per each minute of error in parallax and the formula can be written as

$$\epsilon_H = \pm\, 0{\cdot}92\ \text{km}\left\{\left[(K^2-1)+\frac{r^2}{\rho^2}\right]^2 \cot^2 \varDelta p + \left(4\,\frac{\epsilon_z^2}{\epsilon_{\Delta p}^2}\right)\frac{r^2}{\rho^2}\sin^2 z\right\}^{\frac{1}{2}}$$

$$\simeq \pm\left[(K^2-1)+\frac{r^2}{\rho^2}\right]\cot \varDelta p, \quad \text{km}/' \qquad (25)$$

An estimate of an expected value of cot $\varDelta p$ can be obtained from formula (19) :

$$\cot^2 \varDelta p = \frac{r^2/b^2}{\sin^2 p_1} - 1$$

where $\sin^2 p_1$ would usually be between 1 and 0·1 and r/b is, of course, the ratio of auroral distance to base line length.

X. CONCLUSIONS

The formulae listed in the preceding sections were developed for use with the measuring instrument in conjunction with auroral plates for which the co-ordinates of the centre (a_o, z_o) were known. Nevertheless the formulae have a general applicability.

The use of the measuring instrument is reflected only in the absence of any reference to the way in which the co-ordinates (a, z) for any point in the field of the photographic plate are obtained. They are simply read off in the instrument. The co-ordinates could be similarly obtained by means of Störmer's "nets" (loc. cit.) which can provide the co-ordinates of the centre (a_o, z_o) as well, if at least the location of the centre on the plate is known.

The approximate formulae, as was stated in the introduction, were developed within the limits set by the unusually high precision requirements. The error of the approximate formulae will grow with the increasing length of the base line but the error tolerance, for a long base line, will be much greater. Hence, the formulae are not restricted to the short base line of 31 km, either.

The expressions for the error of the approximate formulae give an immediate answer whether, for a given length of base line and h'-value, the error can be tolerated.

XI. REFERENCES

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AURORAL PARALLACTIC PHOTOGRAPHY

PART 3

METHOD OF DETERMINATION OF AURORAL POSITION

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ABSTRACT

The method is based on the use of a theodolite measuring auroral plates in the light collimated by the camera lens with which the photographs were exposed. Both speed and precision are increased. Equating one lens to another with the help of simple nomographs makes the use of the original camera lens in the collimator unnecessary. A general formula for the error of determination of auroral height is given as well as the reduction formulae.

CONTENTS

I. INTRODUCTION

The method of determination of auroral height and of its distance from the observer from a pair of simultaneous photographs taken at two bases was developed by Störmer and described by him (Störmer 1955) and by Harang (1951). Both authors give also an historical description of the subject.

Briefly, Störmer's method consists of a projection, on tracing paper, of each auroral photograph by means of a photographic enlarger. The focusing is adjusted to give a predetermined scale. Location of the stars appearing in the field is marked on the paper and the lower auroral border is traced by a line. The tracing is then combined, by superimposing, with a co-ordinate grid and the alignment is made so that the grid gives the computed co-ordinates of the stars. The co-ordinates of any point in the field can then be read off from the grid. Further reduction of the parallactic problem is made by means of graphical aids, the computing charts. A large number of co-ordinate grids must be prepared.

Störmer's method has great merits and is briefly outlined above so that reference to it, as a projection method, can be made again in this paper. It is limited, however, to an accuracy of about 6 minutes of arc with which the co-ordinates of the points in the field of the photographic plate can be determined.

The present method is based on the use of an instrument (described in Part 1) for direct measurements of auroral plates thus avoiding the use of grids.

Cameras for taking auroral photographs are not described as the only requirement, in conjunction with the method, is that they are provided with a focal plane graticule indicating the vertical orientation of the plate and its centre (the optical axis). Jacka and Ballantyne (1955) described photo-theodolites which give the co-ordinates (azimuth and zenith distance) for the centre of the plate as well. This feature is desirable but not indispensable provided certain conditions are satisfied which will be discussed in Section IV.

II. NOTATION AND DEFINITIONS

Figure 1 shows the earth assumed spherical of radius ρ and with its centre at O. The cross section is in the plane of the base line and the two bases are denoted by B_1 and B_2.

The relations on the celestial sphere, at infinity, represented by arcs and spherical triangles, are projected back on the earth and are thus shown together with the points separated by finite distances: the bases and the aurora.

Alternatively, one can view Figure 1 as representing the celestial sphere with *all* points at finite distance located at a single point O with B_1 and B_2 indicating the zenith points of the bases.

B_1 is the satellite base serving for selection of (arbitrary) points on the lower auroral border. B_2 is the main base and its auroral photograph serves for final measurements and reduction.

The two bases are separated by a distance (b), the length of the **base line**, with a corresponding angular value, the **base arc,**

$$\beta = b/\rho$$

The chord from B_1 to B_2 is the **base axis** which, produced beyond B_2 to infinity, projects on the celestial sphere at a point, the **base axis point**, with a height h' above (or below) the horizon.

Subscripts 1 and 2 denote values referring to Bases B_1 and B_2 and o refers to the optical axis (plate's centre) for either base. Any other subscript is incidental and is explained in the text. Primes are largely reserved for projected quantities, e.g.:

- P an auroral point with height H above sea level
- P' an orthogonal projection of P on the surface of earth
- r auroral distance from B_2
- r' the great circle arc distance of P' from B_2; orthogonal projection of r onto earth

Horizontal system of co-ordinates:

- z zenith distance of a point on the sky (or on the photographic plate) measured from the zenith along the arc of a vertical circle
- $h = 90 - z$ height above or below the horizon
- z' zenith distance of the base axis point, shown for the main base on Figure 1 on which $z'_1 = z'_2 + \beta$
- $h' = 90 - z'$ is a base constant and its value depends on β and on the difference in height above sea level of the two bases
- a azimuth measured eastwards from true north
- a azimuth measured from a point on the base line lying in the direction of the base axis point in the same sense as true azimuth, i.e. clockwise
- A azimuth of the base line, measured as an angle between the geographical meridian and the base line and not as a bearing from one base to another (i.e. unidirectionally, not contra-directionally).
- a_o, z_o co-ordinates of the centre of the photographic plate (either base)
- $\Delta\mathrm{a} = \mathrm{a} - \mathrm{a}_o$, $\Delta z = z - z_o$ the differences in the auroral field (plate's field).

To make the layout of a base line, in any given orientation, conform with the sign convention of the reduction formulae in Section VII, the choice of the direction to the base axis point assigns the rôle of the two bases B_1 and B_2. Finally, with a and a measured as defined above, the following simple relation, in which A may be negative: $a = \mathrm{a} + A$ must be satisfied. The switch of designation of the bases is mentioned in Section VIII.

Parallactic plane is defined here as a plane containing the base axis and the selected auroral point, $B_1 B_2 P$, and is also shown on Figure 1 by a parallel plane $OO' V_1 V$ in which OO' is parallel to the base axis. Normal plane, $OO' B_2 B_1$, contains the base axis and the centre of earth.

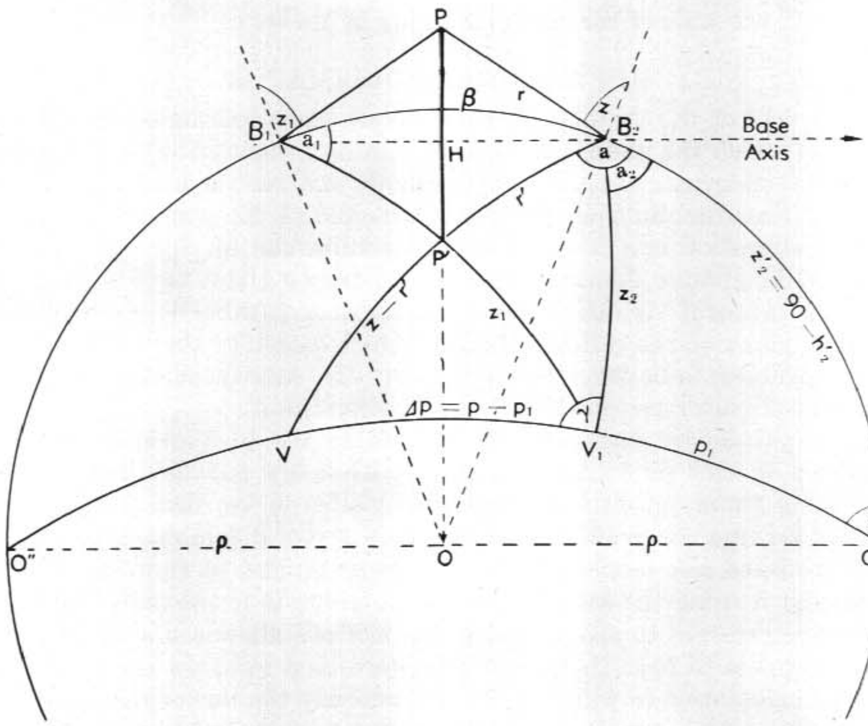

Fig. 1. Notation.

Parallactic co-ordinate system:

- i the inclination of the parallactic plane with respect to the normal plane. Sign of i follows sign of sin a
- p the arc of parallactic displacement measured from the base axis point O'; $0° \leqslant p \leqslant 180°$, always positive. In the auroral field the arcs are referred to simply as parallactic lines or p-lines
- Δp the magnitude of parallactic displacement, a measure for the parallax. $\Delta p = p - p_1$
- γ the direction of parallactic displacement defined by the angle between the vertical circle and the p-line intersecting at a given point in the field; γ is measured on the sky clockwise from the zenith side of the vertical circle

γ' the value of γ projected on the photographic plate

Γ' the projected value of γ referring to $a = a_o$; ($\Delta a = 0$).

Other letters used for specific purposes are explained in the relevant context. The three of greater general significance are repeated below:

θ angular distance from the optical axis (centre) of a point in the field of the photographic plate or on the sky

l linear distance from the centre, corresponding to θ, of a point on the photographic plate

$\sigma = \frac{d\theta}{dl}$ the scale of the optical imaging by the lens.

III. AURORAL COLLIMATOR

The field of the developed photographic plate collimated by the same lens with which the plate was exposed can be measured with a theodolite directly in spherical co-ordinates (azimuth and zenith distance). The instrument thus functions on the same principle as the goniometric method of lens calibration and consists of a theodolite and an auroral collimator described in greater detail in Part 1 and shown there on Figure 1. The adjective "auroral" is due to the presence of a parallactic graticule near the focal plane of the collimating lens which provides the solution of the auroral problem to be described in Section VII. Any theodolite can be used with the collimator provided the magnification is low.

The parallactic graticule, as opposed to the photographic graticule impressed on each photograph, is ruled with a few parallel lines in red to make them stand out among other lines visible in the field. It can be rotated by turning the graduated drum of the auroral collimator to which it is attached and also displaced laterally on its guides, at right angle to the rulings, by turning the knob on the drum which is prominently visible on the Figure referred to above. These two motions allow a line of the parallactic graticule to pass through a predetermined point in the field in the required orientation in which position it becomes the line of parallactic displacement (p-line) passing through the point at an angle γ as defined in Section II and shown on Figure 1. The graduated scale of the rotating drum of the collimator serves for setting angle γ.

The photographic plate is illuminated in translucent light of the condenser which collimates the light of the diffuser. The collimating lens (of the collimator) forms the image of the diffuser at the entrance pupil of the theodolite. The final image of the diffuser appears at the exit pupil of the theodolite just above the eye lens of the eyepiece where the pupil of the observer's eye is situated. The conditions for the Maxwellian View (Longhurst, 1957) are thus satisfied.

Tests were carried out on the optical bench in search of an illuminating system providing a maximum contrast. The Köhler system of illumination (Born and Wolf, 1959), which is often used for microscopes, was tried out. An auxiliary lens of the Köhler system forms the diffuser's image at the focal point of the condenser lens. The condenser collimates the light of the diffuser and at the same time forms an image of the auxiliary lens on the photographic plate which is thus more uniformly illuminated. No definite improvement of contrast was achieved with the Köhler system and a simple condenser with minimized spherical aberration was designed. Spherical aberration seemed to have an effect on contrast, apparently through an "indifferent" location of the diffuser's image at the entrance and exit pupil of the theodolite. Polarizers also produced no improvement, indicating an absence of polarized light in any one plane.

As a matter of interest it is worth reporting that the experimentation with frontal illumination gave promising results, but the conditions required are not possible to fulfil within the design of the collimator and the optics used. Those conditions are:

1. The plate must be turned with the emulsion towards the white background. For this the parallactic graticule with lines ruled on a white surface must be located behind the photographic plate, which is quite possible with a much modified design of the collimator, but
2. the light for frontal illumination must not have a great angle of incidence and its reflection in the glass of the photographic plate must not enter the collimating lens.

As the diameter of the rear lens (or rather of its mounting, since the lens itself could be stopped down) is large, and the back focal length only about 30 mm, this last condition was not possible with the Leitz Summarit lens used in the collimator and would probably be difficult with any objective. To prevent the reflected light entering the lens a stop was placed in the experimental set-up between the lens and the plate with illuminating bulbs on the plate side of the stop around the periphery. With little space available, it produced a near grazing incidence. With other lenses, of longer focal length, the improvement of contrast was very marked when the angle of incidence of the illuminating light was decreased.

IV. ALIGNMENT BETWEEN COLLIMATOR AND THEODOLITE

The photographic plate must be placed in its upright position in the collimator to make the direction of the increasing zenith distance of points in the field agree with the graduations of the vertical circle of the theodolite. The horizontal directions, right and left, are still reversed, unless a positive contact print of the plate is made, and they do not agree with the sense of increasing azimuth graduations on the horizontal circle of the theodolite. However, with the original negative plates, the horizontal circle of the theodolites is set to read zero at the centre of the plate and the azimuth is measured differentially as Δa which was defined in Section II.

The photographic plate is centred on the optical axis of the collimating lens with the aid of the parallactic graticule the centre of which, marked by a cross or circle, is located on the optical axis when the graticule is in the mean position of its lateral movement. In an eccentric position the centre of the graticule appears oscillating when the drum is rotated. Hence, while rotating or swivelling the drum, the graticule is moved laterally until the oscillation stops, which indicates the position of the centre on the optical axis. The plate, with the photographic graticule impressed on it, is then centred with respect to the parallactic graticule using the lateral adjustments of the plate holder.

The centring is not very critical. The error of measurement with the theodolite, ϵ_θ, produced by the (angular) eccentricity error ϵ_{ecc} (malcentring) can be expressed through

$$\frac{\epsilon_\theta}{\epsilon_{ecc}} = \sin^2 \theta$$

in which θ is the angular distance of a point in the field from the centre. Representative values are, e.g. $\epsilon_\theta = 1' \cdot 0$ for $\epsilon_{ecc} = 15'$ and $\theta = 15°$. Derived from simple geometrical considerations the above approximate relation was tested by measurements with deliberately large and known ϵ_{ecc} values. The error in zenith distance, ϵ_z, and the great circle arc equivalent of the error in azimuth, $\epsilon_a \sin z$, are components of ϵ_θ, hence neither can be larger than ϵ_θ itself.

The vertical alignment of the photographic plate, i.e. of the central vertical line of the photographic graticule, is made with respect to the vertical line of the pointing cross of the theodolite using the rotary adjustment of the plate holder.

Assuming, in the first instance, that the co-ordinates of the centre of the plate (a_o, z_o) are known, there remains only to set the theodolite at the reading of z_o and to turn the collimator by its trunnion arms until the centre of the photographic graticule coincides with the pointing cross of the theodolite. The levelling screws of the theodolite are useful in the fine adjustment of this last alignment and the horizontal rotation of the theodolite must be used. The horizontal circle of the theodolite is set to read zero ($\Delta a = 0$) before the measurements begin.

As seen from the above description the theodolite does not need to be levelled in the ordinary sense. It is, in fact, levelled to the horizon of the photographic plate as a consequence of the alignment achieved.

An assumption of the known values (a_o,z_o) for the centre of the plate implies the use of some type of photo-theodolites already referred to in the Introduction. Otherwise the (a_o, z_o)-values must be determined from stars and the mode of camera operation should be such as not to invalidate the advantages of the method described in this paper. The four stages of alignment between the collimator and the theodolite can be done in under five minutes for each of the two plates with a well-constructed collimator. The time to be spent on star reductions and subsequent determination of (a_o, z_o) from the theodolite's pointings on the stars would cancel all efficiency of the method if the above determination is necessary for each pair of auroral plates. Fixed cameras, or cameras the position of which is not frequently changed, would suit the method ideally. At least, the position of the cameras should be kept unchanged for a substantial batch of auroral plates with the pointing that would give a convenient group of stars on the first exposure. The stability of the central vertical line of the focal plane graticule is of particular importance because the determination of its tilt error is much more cumbersome than that of the co-ordinates of the centre.

Measurement of the star field is more convenient with the positive contact prints (on glass plates) which have the same direction of increasing azimuth as the graduations on the horizontal circle of the theodolite. At least two stars are required, more than two are wanted for a check and for certainty. Their equatorial co-ordinates, declination and hour angle, are readily obtainable from the almanacs. Transformation from equatorial to horizontal co-ordinates, azimuth (a) and zenith distance (z), must be computed, or secured otherwise, with atmospheric refraction allowed for in the values of z. Störmer (loc. cit.) used graphical aids. A large list of existing nomograms was compiled by Strassl (1955).

The plate must be centred on the optical axis of the collimator and aligned vertically with respect to the theodolite. The theodolite is set at the reading of (z) of a star nearest the centre which is less affected by the tilt error mentioned before. The star is then brought on the theodolite's pointing cross by means of the trunnion arms of the collimator and the horizontal motion (rotation) of the theodolite. Finally the horizontal circle of the theodolite (always adjustable) is set on the azimuth reading (a) of the star. The theodolite now gives correct readings in both co-ordinates of the star and this will be the case with all stars in the field if there is no tilt error of the vertical line of the photographic graticule. The readings of the centre give the required (a_o,z_o) co-ordinates with the final transformation for convenience in auroral work: $\mathrm{a}_o = a_o - A$.

In the presence of a tilt error the theodolite's readings do not agree with the computed (a, z) values of other stars in the field. Setting the theodolite to read the computed (a, z) values of a remote star the latter is brought to the pointing cross by means of rotation of the plate holder. The process must be repeated until desired agreement is achieved, always readjusting the horizontal circle of the theodolite on the first star, but not on the second. Another star, or stars, in the field serve for a final check.

The numerical value of the tilt error is obtained with the aid of the parallactic graticule. A parallactic line of the parallactic graticule is aligned first to the vertical line of the photographic graticule and then to the vertical line of the theodolite's cross, using the lateral motion of the parallactic graticule and the rotation of the graduated drum. The difference of the drum readings, corresponding to the two successive alignments, is the tilt error.

Reverse process is then used in the collimator-theodolite alignment for the batch of auroral plates referring to the unaltered position of the auroral cameras. After an alignment of the parallactic graticule and the theodolite's pointing cross, the graduated drum is rotated through an angle equal to the tilt error and the photographic plate is aligned to the parallactic graticule.

The particular importance of the photographic graticule's line indicating true vertical direction which remains so in any altered pointing of the auroral camera is clear from the above description. Apart from the more cumbersome collimator-theodolite alignment for auroral plates in the presence of a tilt error, its assured absence makes the (a_o, z_o) determination particularly simple when the camera position is altered. The positive contact prints of the plates become quite redundant. With the known (a, z) of a star near the centre, the difference of the readings of the theodolite when it is pointed on the star and then on the centre gives the Δa, Δz, values from which we have a_o, z_o directly. Aligning the collimator to the theodolite with the z_o-value (and zero for azimuth reading) we check Δa, Δz, for a second star which may be near the centre of the plate as well. The second star is, however, still important because a possible computing error in (a, z) values of the first star goes fully and undetected into the values of (a_o, z_o).

It may be noted, in passing, that three simultaneous equations would give the solution for a_o, z_o and the numerical value of the tilt error from non-repetitive pointings on stars in the field instead of successive alignments and pointings. Their introduction would be, however, not worth while in our case because experience has shown a great disparity in time spent at the instrument and at computing, the latter being always much longer. We would merely lengthen the total time required for the sake of the more elegant method.

V. FIELD CORRECTIONS

To achieve full precision of measurements requires fulfilment of the condition that the same objective must be used with which the plate was exposed and that it must be in the same orientation with respect to the plate in the collimator as it was in the camera. The latter condition is invalidated by the design of the collimator because the parallactic graticule is rotated by the drum which carries the objective. The errors introduced by the arbitrary orientation of the objective are due to asymmetrical properties of field distortion, depending on direction. It would, however, require a very bad lens to produce errors of this type exceeding the limits of tolerance for auroral work.

Substitution of one lens by another with the identical maker's specifications can produce, however, much greater errors. Lenses are generally not identical in their field characteristics and in their focal lengths and only an accurate lens calibration can give an indication whether the lenses can be considered interchangeable or not.

It is on the whole simpler to construct two collimators and measure the plates from each base with their own objectives. There is, of course, no need for two theodolites.

Alternatively, one can design an adaptor and change the lenses when the plates are changed, so that a correct lens is used with a plate. This calls however for very high precision of engineering, as the lens must be centred with respect to the centre of the drum **within** the adaptor and the latter must be so precisely located each time it is inserted into the drum as not to alter the achieved centring of the lens. One can estimate the effect of eccentricity by simple geometrical considerations which show that the ratio of linear eccentricity (e) and the focal length (f) is equal to the angular eccentricity ϵ expressed in radians. The latter is equal to an error produced by 90° rotation of the drum. Hence:

$$\frac{e}{f} = \epsilon \text{ in radians, i.e. } e = \frac{\epsilon'}{3438'} f$$

An accepted maximum of ϵ' (in minutes of arc) can, in conjunction with the above formula, be nearly doubled because the rotation of the drum is generally much less than 90° during the measure. On this basis one can say that the tolerance of linear eccentricity is $\frac{1}{2}\ 10^{-3}$ of the focal length (f) for each one minute of ϵ'. This is reasonably easily achieved in the design of the adjustable lens mounting but it might be much more difficult with an adaptor, owing to its locking mechanism.

The second alternative consists in the use of a single lens to which corrections are applied when a plate taken with another lens is measured. These field corrections are derived from comparison of an accurate lens calibration of the two lenses which gives a correlation between the angular values (θ) and the corresponding linear values (l) on the photographic plate.

For a given linear distance from the optical axis (l) the two lenses give values of θ differing by $\Delta\theta$ which is a correction to the angular distance from the optical axis of a point on the photographic plate if that plate was exposed with one lens and measured with another. However, the distances from the optical axis are not measured on the auroral plates and we must split $\Delta\theta$ into the corresponding corrections in azimuth and zenith distance. Let us consider a spherical triangle formed on the celestial sphere by the great circle arcs between the Optical Axis Point, Zenith Point and an arbitrary point corresponding to a point on the photographic plate. Its sides are z_o, z and θ and we shall call the angle at the point, formed by arcs z and θ, angle U. U lies opposite z_o. With the variation of θ only z and Δa can vary, not z_o, and a variation of Δa is equal to the variation of a itself. Differentiations of any suitable basic formulae give, after transformation, the same result:

$$\frac{dz}{d\theta} = \cos U \qquad \frac{d\mathrm{a}}{d\theta} = -\frac{\sin U}{\sin z}$$

One obtains a seemingly identical result from a right angled plane triangle with the hypotenuse $\Delta\theta$ and the sides Δz and Δa sin z, the last being the great circle arc equivalent of Δa, but in the above formula the value of U is spherical and in the plane triangle it is the projection of U. The negative sign in the second formula arises from a sign convention for U which was necessary during the expansion. We can now express U through formulae containing only known elements, i.e. z_o, z and Δa and, inevitably, sin θ. Furthermore, the differentials are expressed directly as corrections $d\theta = C_\theta$, $dz = C_z$ and $d\mathrm{a} = C_a$. Suitable expressions are then:

$$\frac{C_z}{C_\theta} = \cos U = \frac{\cos z_o \sin z - \sin z_o \cos z \cos \Delta \mathrm{a}}{\sin \theta}$$

$$\frac{C_a \sin z}{C_\theta} = -\sin U = +\frac{\sin \Delta \mathrm{a} \sin z_o}{\sin \theta}$$

The correction to the distance from the optical axis C_θ and the sin θ in the above formulae are correlated, both increasing numerically, and we may assume, in the first approximation, that the ratio $(C_\theta/\sin\theta) \simeq$ const. The above assumption simplifies the formulae considerably because C_θ is not known until θ is computed from Δa, z and z_o. What is more, the expressions for C_z and C_a become easily nomographable.

The nomographs are given below in the form of determinants in which the first two columns are, respecively, parametric equations in x and y. The first determinant, following the analytical expression from which it was derived, is standard. It does not necessarily give a workable nomograph but forms a basis for any desired projective transformations, one of which is given by the second determinant. The latter becomes constructional determinant after scale factors are applied to the columns for x and for y, depending on the size of the nomograph wanted and on the units of scale used (inches or millimetres). We obtain the result for correction in zenith distance as

$$\zeta + \sin z_o \cos z \cos \Delta \mathrm{a} - \cos z_o \sin z = \begin{vmatrix} \frac{1}{1+\sin z_o \cos z} & \frac{\cos z_o \sin z}{1 + \sin z_o \cos z} & 1 \\ 0 & \cos \Delta \mathrm{a} & 1 \\ 1 & \zeta & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1 - 2\sin \Delta z}{1 + 4 \sin z_o \cos z} & \frac{4 \sin \Delta z}{1 + 4 \sin z_o \cos z} & 1 \\ \frac{1}{2}(1 - \cos \Delta \mathrm{a}) & (1 - \cos \Delta \mathrm{a}) & 1 \\ (1 - 2\zeta) & 4\zeta & 1 \end{vmatrix} = 0$$

where $\Delta z = z - z_o$ and $\zeta = C_z \cdot \frac{\sin\theta}{C_\theta} = \frac{C_z}{\text{const.}}$

ζ is used only for plotting but the resulting graduations are numbered directly as C_z. The nomograph made from the second determinant is a kind of multigrid with lines $z_o = \text{const.}$ graduated in Δz. It is slanted and it appears to be better that way. If preference is shown for vertical scales for Δa and ζ the second column, with a factor $\frac{1}{2}$, must be added to the first.

The expression for azimuth correction, C_a, is more conveniently shown by a double alignment nomograph instead of a multigrid and requires two determinants..

Substituting $\alpha = \frac{C_a}{C_\theta} \sin\theta = \frac{C_a}{\text{const.}}$ we can write:

$$\sin \Delta a \sin z_o = k = \alpha \sin z$$

$$\begin{vmatrix} 0 & \sin \Delta a & 1 \\ \frac{1}{1+\sin z_o} & 0 & 1 \\ 1 & -k & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} 0 & \alpha & 1 \\ \frac{1}{1+\sin z} & 0 & 1 \\ 1 & -k & 1 \end{vmatrix} = 0$$

k is an auxiliary variable giving an intermediate scale which connects Δa, z_o with α, z; its line and graduations are common to both parts of the nomograph. The first row in the two determinants represents a single line in the nomograph with a double scale (on both sides) one for Δa and one for α. The graduations of α are numbered directly in C_a just as the graduations of $\sin \Delta a$ are numbered in Δa. The middle row represents a single line with a scale common to both z_o and z. There remain only scale factors to be applied to the columns of the determinants to make them ready for computing.

The method of applying corrections which equate the field of one lens to that of another suits particularly a big programme planned for more than a year and probably involving the use of several cameras. One lens can be selected to be fitted permanently in the collimator. It must be accurately calibrated and the same applies to every lens to be sent out to the bases. A pair of nomographs is constructed for each lens if its calibration shows that field corrections are required. There is no need to wait until the end of the programme and the plates can be measured while the cameras are still in operation.

Calibration of the two Leitz Summarit lenses with which auroral plates were secured showed maximum C_θ of $2'\cdot 5$ for a $40°$ field or about three times the tolerable error (taken at an abnormally low value in this case). Analysis of calibration curves gave $8' < (C_\theta / \sin\theta) < 10'$ within the measurable field and a constant value of $9'\cdot 0$ was adopted. A note on the nomographs was, however, added advising a 10 per cent variation of the obtained corrections for points near the optical axis and near the margin of the field.

The ratio $(C_\theta / \sin\theta)$ is one of the field characteristics of the lenses and will be discussed in the next section. When it cannot be assumed constant at all a construction of 4 nomographs, using angles U, would be necessary which is admittedly cumbersome both in "design" and in use.

VI. FIELD CHARACTERISTICS

This section covers wider aspects of field distortion, such as scale characteristics of a lens, and is important for the analysis of lens calibration for the field corrections of the preceding section. It has also a bearing on the projection method with which the present method is compared in the conclusions (Section IX). Before proceeding with the subject certain optical principles require reiteration in a form suitable for the present purpose.

By general definition distortion is caused by variable magnification within the field, increasing or decreasing outwards and producing, respectively, pincushion or barrel distortion.

A lens calibration gives a correspondence between the linear (l) and the angular (θ) values for points on the photographic plate. In a field with constant magnification, distortionless field, the values of (l) are proportional to the values of $\tan\theta$.

$$l = f \tan\theta$$

and the proportionality factor f is the focal length of the lens. The expression above shows that a field without distortion is gnomonic in which all geodesics are imaged as straight lines.

The presence of distortion can be inferred from comparison of calibration values with those obtained by the gnomonic formula; we differentiate the latter and obtain the scale (σ), the definition of distortion, and the numerical value of the focal length as:

$$\sigma = \frac{d\theta}{dl} \lesseqgtr \frac{\cos^2\theta}{f} \begin{cases} \text{pincushion distortion} \\ \text{gnomonic imaging} \\ \text{barrel distortion} \end{cases}$$

Thus the type of distortion depends on the scale being smaller or larger than the gnomonic scale. For $\theta = 0, \frac{d\theta}{dl} = \frac{1}{f}$ in units of one radian. The above well known relation, giving the value of (f) from the known scale, is fundamental and applicable to lenses with any characteristics because there is no distortion in the immediate proximity to the optical axis.

It is worth noting that a constant magnification implies a variable scale, decreasing outwards with $\cos^2\theta$, while a lens giving a constant scale, equal to $1/f$ throughout the field, produces decreasing magnification and is affected by barrel distortion with the consequent parabolic imaging of the geodesics (Conrady 1957).

Field distortion can be expressed centesimally as

$$100\left(1 - \frac{l}{l_G}\right)$$

where l is the calibration value, l_G the gnomonic value, both referring to the same value of θ.

The above representation of distortion is shown on Figure 2 for a

Leitz Summarit lens. Distortion for a hypothetical constant scale lens, in which the ratio of l/l_G is equal to $\theta/\tan\theta$, is added for comparison. The same amount of barrel distortion occurs with the latter lens about 10° earlier than in the field of the calibrated lens.

An assumption that the geodesics are imaged as straight lines is generally tolerable for a distortion under one per cent within the measurable field. It is, of course, the angle γ, or Γ' of Section VII, which is largely affected.

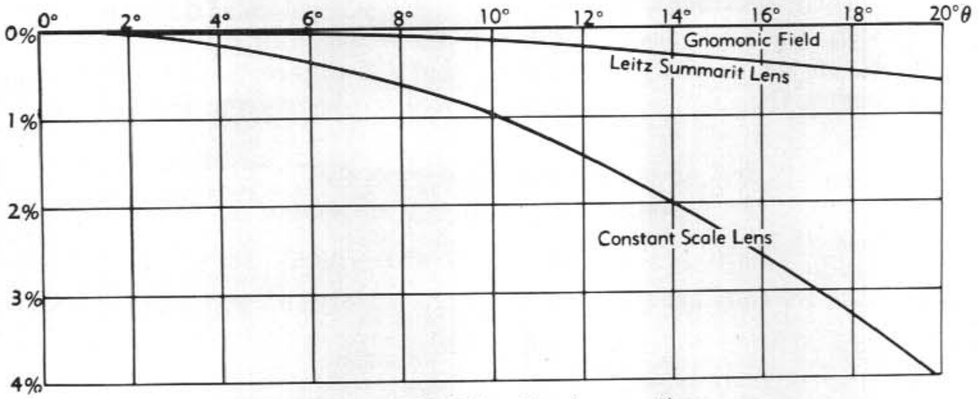

FIG. 2. Field Distortion (per cent).

The distortion curve of the Leitz Summarit lens is affected by the focal plane graticule (0·1 inch thick glass plate) with which the lens was calibrated. A ray tracing of the graticule, in conjunction with a distortionless, gnomonic field, reveals a positive lateral displacement of the images, producing a pincushion effect. The magnitude of the displacements is small and they do not follow the l^3 law of the field distortion, increasing actually with less than l^2. With reference to Figure 2, at a distance of 14° from the optical axis, the comparative values are:

$-0{\cdot}1\%$ pincushion effect of the graticule
$+0{\cdot}3\%$ barrel distortion of the calibrated lens with the graticule
$+2{\cdot}0\%$ barrel distortion of the constant scale lens.

All other aberrations are produced by the graticule's glass plate but, as with the displacements, their amount is small and of little significance. Nevertheless it is advisable to restrict the glass thickness to a bare minimum acceptable to the graticule's maker. In some cases the presence of the focal plane graticule can be beneficial It acts as a field flattener if a slight positive curvature is present in the field of the lens. Conversely, it would of course introduce a slight negative field curvature if the initial field is perfectly flat, in which case the magnitude of the lateral displacements is very sensitive to focusing and their sign can be reversed near the margin of the field. In all cases the graticule will counter the barrel distortion of the lens up to a certain distance from the optical axis.

In the preceding section, on the field corrections, an assumption was made that the ratio

$$\frac{C_\theta}{\sin\theta} \simeq \text{const.}$$

in which C_θ was the required correction if a plate in the collimator is measured through an objective with which it was not exposed.

Calibration of the two lenses gives different values of θ for the same values of l, hence the ratios $C_\theta/\sin\theta$ are immediately available. They will not be nearly constant if the distortion characteristics of the two lenses are grossly different, which is unlikely for lenses with identical specifications. If, on the contrary, we assume that the difference lies only in the focal length, then for distortionless lenses we have from $l = f_1 \tan\theta_1 = f_2 \tan\theta_2$:

$$\frac{\Delta f}{f} = \frac{\Delta \tan\theta}{\tan\theta} = \frac{\Delta\theta \sec^2\theta}{\tan\theta} = \frac{C_\theta}{\sin\theta\cos\theta} = \text{const.}$$

hence $\frac{C_\theta}{\sin\theta} = \text{const.} \times \cos\theta$, decreasing with θ by 6% for $\theta = 20°$. On the other hand, for constant scale lenses (barrel distortion) we have:

$$\frac{\Delta f}{f} = \frac{\Delta\theta}{\theta} = \frac{C_\theta}{\theta} = \text{const.}$$

and $\frac{C_\theta}{\sin\theta} = \text{const.} \times \frac{\theta}{\sin\theta}$ increasing with θ by 2% for $\theta = 20°$. For lenses with characteristics one third of the way from constant scale to distortionless field the ratio $C_\theta/\sin\theta$ would be virtually constant. The difference in distortion characteristics, however slight, will affect the values of the ratio, but the values of C_θ would still be increasing numerically (as does $\sin\theta$) because there is no distortion near the optical axis. Owing to the small amount of the field corrections a high accuracy in their determination is not required and the nomographic method of the preceding section should be valid.

The pincushion type of distortion is most unfavourable. It is increased, however slightly, by the use of a focal plane graticule. It has a stronger effect on the variability of the ratio discussed above and has neither the advantage of a near constant scale of barrel distortion nor the advantage of the linearity of the geodesics in distortionless lenses. The distortion characteristics of the lenses is not the least point to bear in mind for parallactic auroral work.

With respect to other features of the lenses it has been pointed out by Störmer (1955) and particularly by Harang (1951) that while the high aperture ratio, i.e. small f-factor, is desirable, the total glass thickness of the objective should be taken into account as well. A high degree of correction in an objective, usually accompanied by a large number of components with the resulting increase in total glass thickness, is not a necessity in the photography of diffuse auroral forms. The light gathering power (the *f*-factor) of a lens should not take complete precedence over the light transmitting factor which suffers in the presence of thick glass particularly in the lower spectral region. The balance of both factors is the best choice.

VII. MEASUREMENT AND REDUCTION

Reference is made here to Part 2 for a more detailed discussion of various aspects of the auroral problem which is stated only briefly in the presented paper. For derivation of formulae used in this section the reader is also referred to Part 2 which further contains formulae for the radius of curvature of the earth, ρ, for the main base. The value of ρ, supposed here to be known, is one of the six Base Constants already defined in Section II: ρ, β, A_1, A_2, h'_1, h'_2.

The length of the base line (b) equal to the geodetic distance between the two bases and the orientation of the base line, defined through its azimuth values A_1 and A_2, are obtained either by surveying or by computing from the data of the geographical location of the bases. The great circle arc value is then $\beta = b/\rho$.

With the values of (b) and (β) already defined and denoting by ΔH the difference in altitude of the two bases above sea level in the sense satellite base (B_1) *minus* main base (B_2), the height h' of the Base Axis Point above (or below) the horizon can be computed from

$$\tan h' = \pm \left(\tan \frac{\beta}{2} \mp \frac{\Delta H/\rho}{\sin \beta}\right),$$

or

$$h' \doteq \pm \left(\tfrac{1}{2} \frac{b}{\rho} \mp \frac{\Delta H}{b}\right)$$

The upper and lower signs in the above formula refer to the h'-value for the main base (h'_2) and for the satellite base (h'_1) respectively.

As seen from Figure 1, when the latter is viewed as representing the celestial sphere with points B_1 and B_2 denoting the zenith points of the bases, a point at infinity such as V_1 is defined through different (a, z) values of the horizontal co-ordinate system referring to the two bases. This is due to the presence of two distinct horizontal co-ordinate systems, one for each base. The difference in the values of a and the values of z is the co-ordinate transformation from one base to another for one and the same point at infinity.

The parallactic co-ordinate system, as defined in Section II, is common to both bases and the values of (p) and (i) for a point at infinity (V_1) are the same for either base. If, however, the point in question is not at infinity, its projection on the sky will be parallactically displaced along the p-line (to V on Figure 1) and its co-ordinates as measured from B_2 will be $p \neq p_1$, but with unchanged value of (i), i.e. $i = \text{const.}$ V_1 and V lying on the same line of parallactic displacement represent a projection on the sky of one and the same point at finite distance, as measured from the two bases, a point at the lower auroral border arbitrarily selected from the base B_1. Conversely, an intersection of the line of parallactic displacement, defined through $i = \text{const.}$, with the lower auroral border re-identifies for base B_2 the auroral point arbitrarily selected at base B_1. This method of reidentification of auroral points was first used by Störmer (loc. cit.).

Referring again to Figure 1 the selected auroral point is defined by the co-ordinates (a_1, z_1) from which the values of (i) and (p_1) can be computed from the spherical triangle $B_1\,O'\,V_1$. The co-ordinate transformation, to obtain the values of a_2 and z_2, can be computed from the spherical triangle $B_1\,B_2\,V_1$. The only advantage of such transformation is that it gives a point on the sky (V_1 in this case) which definitely lies on the correct p-line which in turn is defined by the original value of i. We can in fact compute the co-ordinate transformation from the triangle $B_2\,O'\,V_1$ and we can also find a pair of values (a, z) from the same triangle in which V_1 is substituted by an arbitrary point on the p-line. For any point on the p-line the value of angle γ (as shown on Figure 1) can be computed and in the field of the photographic plate it gives the orientation of the p-line with respect to the vertical circle. One of the lines on the parallactic graticule can be made to pass through the point (a, z) with the slope equal to angle γ and the intersection of the line with the lower auroral border indicates the point for which the final theodolite reading must be made. This is the auroral point originally selected from the base B_1.

We can consider now the auroral field on the photographic plate, supposed to be taken from the main base, shown on Figure 3. The photographic graticule consists, in this case, of two circles and cross lines ruled with 0·1 inch spacing. For our purpose, only the central vertical line and an indication of the centre of the plate (the optical axis of the camera) are necessary though other lines have proven useful in the test measures. They signify, gnomonically (i.e. in distortionless field), the great circle arcs converging towards their pole which is located 90° from the centre (a fact utilized in investigation of γ-angles). In the absence of tilt error, as defined in Section IV, the central vertical line passes through the zenith point and represents the central vertical circle of the horizontal co-ordinate system. The scale of the photograph is about $2^\circ\!\cdot 8$ per division of the photographic graticule.

A single line of the parallactic graticule, the p-line, is shown passing through the pointing cross of the theodolite across the field. The pointing cross is of course centred in the field of view of the theodolite which is about 20° (7 divisions). Its vertical line goes actually through the whole field and indicates the true direction of the vertical circle for any point on the photograph. It is thus shown slanted with respect to the lines of the photographic graticule. Figure 3 is a positive contact print of the original negative plate, hence the sense in which angles γ are measured is the same as on the sky: clockwise for positive values. Angle γ is easily computed

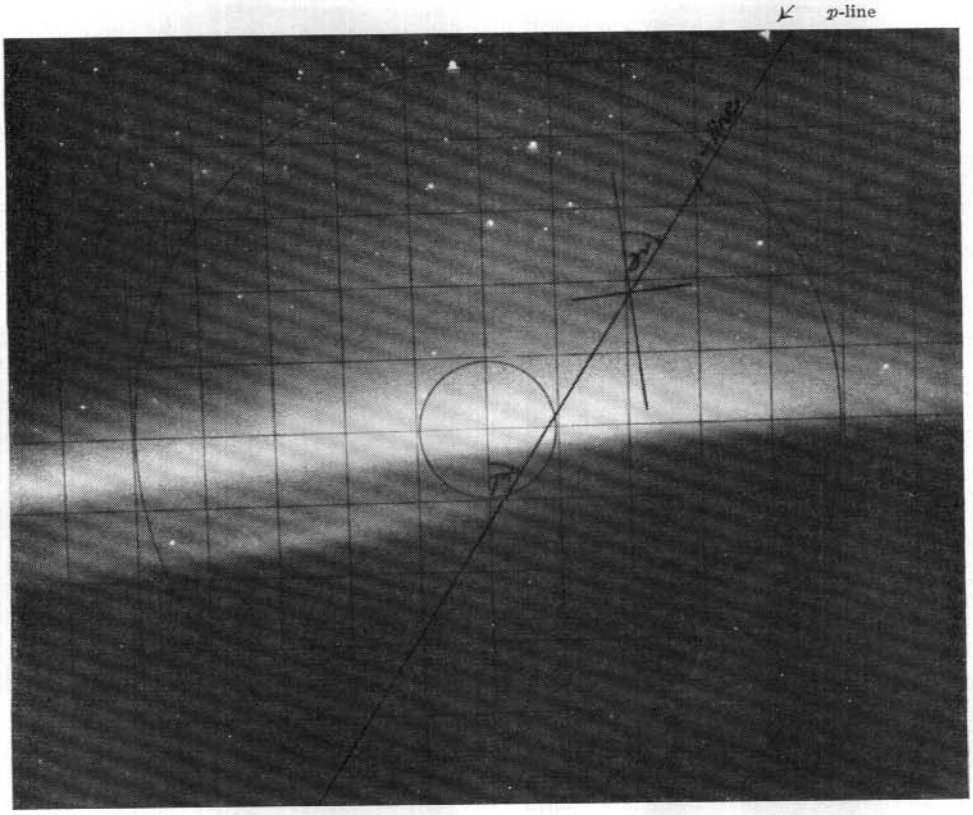

FIG. 3. Field of Auroral Photographic Plate.

but not its gnomonically projected value γ' (assuming a distortionless field) the computation of which is very cumbersome. The difference $|\gamma - \gamma'|$ can reach 2° for a point at a distance of 15° from the centre and can be still 1° for a point 11° from the centre. The use of the value of γ, instead of γ', is thus restricted to points near the optical axis or at least so near to the lower auroral border that a large angular error in γ does not affect the value of the computed parallax (Δp). However, the value of γ may be computed for any point on the p-line ($i = \text{const.}$), irrespective of the actual pointing of the theodolite, and we select the intersection of the p-line with the central vertical for which the gnomonically projected value, which we denote by Γ', can be readily computed. The azimuth value of this intersection, a_o, is known and is constant for all selected auroral points for one plate and computation of the zenith distance z_Γ is easy. The value of Γ', computed gnomonically, is still affected by field distortion but if the latter is under 1 per cent the error, which would be less than $0^\circ{\cdot}2$, can be ignored as was already stated in Section VI. For greater distortion the p-lines on the parallactic graticule must be curved or the usable field restricted.

The procedure of measurement, as opposed to computing, is very short. The alignment between the collimator and the theodolite was described in Section IV. For each auroral point (a_1, z_1) selected on the first plate the value of (i) is computed. The theodolite is set at the reading (a, z) obtained either from co-ordinate transformation or from the value of (i). Allowing for the index error the graduated drum of the collimator is turned to read the angle Γ'. Using the lateral movement of the parallactic graticule, the nearest p-line is made to pass through the pointing cross of the theodolite. The theodolite is then moved to the point of intersection of the p-line with the lower auroral border and the final (a, z) reading is recorded. If γ-angle is used, the drum reading corresponding to an alignment between a line of the parallactic graticule and the theodolite's vertical (at the point a, z) is increased by the value of γ.

The number of points selected is a matter of personal choice and economy of time is dictated entirely by computing and not by measurement. The latter is in fact so short that certain elaborations were made, such as multiple pointings on the estimated intersection of the p-line with diffuse auroral border. An extra pointing of the theodolite was sometimes made on the intersection of the p-line with the central vertical (which is not always within the field) for a comparison of the theodolite's reading with the computed z_Γ value—a very reassuring test.

Transformation of co-ordinates by the standard formulae of spherical trigonometry

$$\begin{aligned} \sin z_2 \cos a_2 &= \cos\beta \cos a_1 \sin z_1 - \sin\beta \cos z_1 \\ \sin z_2 \sin a_2 &= \sin z_1 \sin a_1 \end{aligned} \qquad (1)$$

is too cumbersome but we can approximate the transformation as

$$a_2 = a_1 + \delta a \text{ and } z_2 = z_1 + \delta z \text{ where}$$

$$\delta a \simeq \beta \sin a_1 \cot z_1 \qquad (2)$$

$$\delta z \simeq -\beta \cos a_1 \qquad (3)$$

A nomograph can be easily constructed for formula (2), while (3) is best represented by a small critical table.

The value of $i = \text{const.}$ for each selected auroral point is computed from

$$\sin a \cot i = \cos h' \cot z - \sin h' \cos a \simeq \cot z - h' \cos a \qquad (4)$$

The subscript (1) is omitted from a, z, h' in the above formula because it has other applications referring to B_2 and because the value of i can be computed with a_2, z_2, h'_2 as well if the co-ordinate transformation was used.

To compute a point (a, z) lying on the p-line we select a value of a and compute z by (4) with the value of $\cot i$ already known and using, of course, h'_2-value. To compute z_Γ formula (4) is used again with the value a_o which is constant for the whole second plate (referring to B_2). The value of Γ' is computed from

$$\tan \Gamma' \simeq \tan a_o \frac{\cos (z_\Gamma - z_o)}{\cos (z_\Gamma + M_o)} \qquad (5)$$

while

$$\cot \gamma = \frac{\cos a \cos z - \tan h' \sin z}{\sin a} \simeq \cot a \cos (z + M) \tag{6}$$

if γ-angle is used. The choice of an a-value for the (a, z) point is sometimes made easier if $z_{\Gamma'}$ and Γ' are computed first. Finally we need the values of p for initial pointing from B_1 and the final pointing from B_2:

$$\cos p = \sin h' \cos z + \cos h' \sin z \cos a \simeq \cos a \sin (z + M) \tag{7}$$

The auxiliary function M is common to formulae (5), (6) and (7) and is defined as

$$\tan M = h' \sec a \simeq M \tag{8}$$

Owing to the slow variation of M with a it is conveniently tabulated as a critical table.

The approximation error of the formulae will be dealt with in the next section because it depends on the length of the base line as does the error in auroral height.

The simplicity of the final reduction formulae does not warrant the introduction of approximations.

$$\frac{r}{\rho} = \frac{\beta \sin p_1}{\sin (p - p_1)} \tag{9}$$

r is the distance to the auroral point and p is the value derived from the final pointing, both referring to B_2. Auroral height H is computed from

$$\left(1 + \frac{H}{\rho}\right) = \left(2\frac{r}{\rho} \cos z + \frac{r^2}{\rho^2} + 1\right)^{\frac{1}{2}} \tag{10}$$

where z is the zenith distance of the final pointing. The distance (r') to the projected auroral point (P'), if wanted, is given by

$$\tan r' = \frac{\frac{r}{\rho} \sin z}{1 + \frac{r}{\rho} \cos z} \tag{11}$$

VIII. DISCUSSION OF ERRORS

It has been shown in Part 2 that the error in auroral height H can be expressed as

$$\epsilon_H = \pm\, 0{\cdot}92\,\mathrm{km}\left\{\left[(K^2 - 1) + \frac{r^2}{\rho^2}\right]^2 \cot^2 \varDelta p + 4\frac{\epsilon_z^2}{\epsilon_p^2}\,\frac{r^2}{\rho^2} \sin^2 z\right\}^{\frac{1}{2}}$$

$$\simeq \pm \left[(K^2 - 1) + \frac{r^2}{\rho^2}\right] \cdot \left(\frac{r^2/b^2}{\sin^2 p_1} - 1\right)^{\frac{1}{2}} \quad \mathrm{km}/'$$

which gives a result in km for each minute of arc error of measurement along the p-line (ϵ_p). z refers to the final pointing from B_2 on the auroral border and its error, ϵ_z, is generally smaller than the error along the p-line,

ϵ_p. The ratio of ϵ_z/ϵ_p depends on the slope of the p-line and is roughly equal to cos γ. $\Delta p = (p - p_1)$ as already defined and, using formula (9),

$$\cot^2 \Delta p = \frac{r^2/b^2}{\sin^2 p_1} - 1 \quad .$$

$K = \left(1 + \frac{H}{\rho}\right)$ and

$$(K^2 - 1) = 0{\cdot}033 \pm 0{\cdot}003$$
$$\text{for } H = 105 \text{ km} \pm 10 \text{ km}$$

allowing $(K^2 - 1)$ to be taken as constant while the range of r^2/ρ^2 is

$$2 \cdot 10^{-4} < \frac{r^2}{\rho^2} < 2 \cdot 10^{-2}$$

The factor to $\cot^2 \Delta p$ is thus always very small and the error, ϵ_H, depends entirely on the parallax which, in turn, depends on the ratio of auroral distance to base line length, r/b.

The second term of the exact expression for ϵ_H is of little significance for base lines under 100 km and is therefore dropped from the approximate formula with only a slight increase in the kilometre factor from 0·92 to 1.

For long base lines, well over 100 km, for which the first term can be very small and even vanish, the second term becomes significant and it shows that the base nearer to the aurora should be selected as the main base, particularly when the ratio of the auroral distance from each base exceeds two.

Proximity of the aurora to B_2 and B_1 might alternate, requiring a switch of designation of the bases with consequent reversal of sign in formulae (2) and (3) and in Δp of formula (9). Transformation of co-ordinates, however, does not apply to long base lines because the value of Δp might exceed the length of the whole field; this can also happen in cases of overhead aurora in conjunction with a base line of only 30-40 km.

Incidentally, this switching the rôle of the bases can be used to check the solution. The final pointing (a, z) from B_2, now denoted as (a_2, z_2) is taken as the selected point. The already computed value of p is denoted by p_2 and the value of i checked (it should not alter). A pointing on the p-line from B_1 is obtained by (2) and (3) with signs reversed or by (4) with h'_1 value, which is also to be used for computing $z_{\Gamma'}$ and Γ'. The final pointing on the auroral border from B_1 gives the p-value and $\Delta p = (p_2 - p)$ from which the remaining reduction is made.

The approximate formulae were developed for a 31 km baseline with $h' < 10'$ and the error of approximation was kept under 1′. For longer base lines the approximation error grows but so, generally, does the error tolerance.

Formulae (2) for δa and (3) for δz of the co-ordinate transformation are exact along the base line (a = 0) and (2) is also exact for a = 90°. Maximum error in δa occurs for a = 45° (with corresponding values in the other 3 quadrants) and in δz for a = 90° with the error increasing towards the zenith in both cases. Values of maximum errors in δa, as its great circle arc equivalent, and in δz, for a 31 km base line, are given below:

	a = 45°	a = 90°
z	$\epsilon_{\delta a} . \sin z$	$\epsilon_{\delta z}$
1°	2′·0	2′·2
2	1·1	1·2
3	0·8	0·8
4	0·5	0·5

For other base lines a few error values can be computed from comparison of results obtained from formula (1) with those obtained by (2) and (3). Where the error exceeds the pre-set error tolerance the corresponding small range of z-values may be excluded from auroral measurements or a small correction table can be constructed.

The maximum error of the approximate expression in formula (4) reaches the value of only $\frac{1}{4}h'^2$ in the region of $i = 45°$ or $z = 45°$ depending on whether the formula is used for computing i or z (or z_Γ). The value of $\frac{1}{4}h'^2$ corresponds to one minute of arc for $h' = 2°$ and varies unidirectionally with the square of h'.

The approximation error for γ and Γ' is the same because the transformation of Γ to its projected value Γ' is exact. The error is zero along the base line and at a maximum for a = 90° (cot a = 0) where its value is: $|\epsilon_\gamma|_{\max} = h' \sin z$. If this maximum error exceeds the pre-set error tolerance, a few values computed from the exact and the approximate formula (6) close to a = 90° will indicate the region to be either excluded or corrected by a separate table. Some points must, however, be borne in mind. Firstly, the error tolerance of an angle is much greater than the error tolerance of an arc and the two are equated if the great circle arc equivalent is taken for the former. The above implies for γ-error: $\epsilon_\gamma \sin \delta p$, where δp is the length of the p-arc between the theodolite's pointing, as shown on Figure 3, and the final pointing on the auroral border which can be made by formula (4) as much smaller than the parallax (Δp) as desired. Secondly, for a = 90° or anywhere near that value, the intersection of the p-line with the auroral border occurs generally at near grazing incidence. Tabulation of M, by formula (8), becomes also very inconvenient near a = 90° and in conjunction with formula (5) the camera pointing (a_o) and be restricted to the value of a for which M was tabulated.

As in the preceding case the approximation error of p in formula (7) is zero along the base line and maximum for cos a = 0 where it reaches the value $|\epsilon_p|_{\max} = h' \cos z$. The p-error is thus more troublesome than was the case with angle γ, firstly because it is an arc error and can be given no latitude with respect to error tolerance, secondly because its factor is $\cos z$ and not $\sin z$. It drops, however, very rapidly to about one eighth of the maximum value at a = 89° instead of 90°. By the same procedure as was described above the limits of exclusion or correction can be established. It is, of course, convenient to have the same limits for M, p and Γ' (or γ).

IX. CONCLUSIONS

The main feature of the method described in this paper is the combination of speed with precision of measurements achieved by the use of a theodolite and the parallactic graticule in the collimator. The significant error in the estimated location of the lower auroral border, the pointing in the present case, is not further aggravated by the error in the "reading" of the pointing. The former can be put, arbitrarily, at

$$10' \pm 5'$$

while the latter is a fraction of a minute of arc.

By comparison, the reading error of the projection method, outlined in the introduction, may be one half or more of the error in the estimated location of the lower auroral border which is traced by a pencil line. The re-identified auroral points lie along the pencil line which, owing to its sharpness, gives consistent results in the computed H-values; but the error is still the same, as arbitrarily defined in the first paragraph above, only it is hidden in the line itself and is thus more or less systematic.

In the present method, on the contrary, the errors of individual pointings along the diffuse auroral border are fully reflected in the computed H-values. The apparent drawback of the absence of the pencil line has the inherent advantage of giving an awareness of the errors instead of a false sense of security induced by the consistency of the results, and the magnitude of the error in the average H-value for one pair of plates or the relative merits of the results for different pairs of plates are immediately known. The fortuitous errors can be reduced, when necessary, by multiple pointings, as mentioned in Section VII, and by a technique in which the eye is scanning the border while the pointing is made, without introduction of a systematic error which is always inherent in the fixed bias of the drawn pencil line.

As regards contrast the present method is at a disadvantage as compared with the projection method, but probably not enough has yet been done, since Störmer's pioneering work, in selecting optics with contrast effect in mind and with photographing techniques such as pre-sensitizing the plates. As the method stands at present good plates produce good results but those of poorer quality require contact prints on plates with contrast emulsion.

The collimator can be adapted for use with films which are sometimes preferred, particularly with automatic cameras, or contact prints on glass plates can be made from selected frames of the film. In the latter case it is worth considering making the contact prints on contrast paper and using a frontal illumination, instead of a condenser, thus achieving an even better contrast than is possible with the projection method.

With reference to Part 1, certain points that gave trouble either in the design stage or in the later use of the instrument deserve mention as a suggested scope for improvements.

The pointing cross of the theodolite must consist of double lines which would speed up their alignment with the single lines of the photographic and the parallactic graticules. For good visibility even with the faintest optimum illumination by the condenser (adjustable by means of a variometer) the lines need not be more than 6μ thick and in any case must be less than 10μ (0·0004 inch).

The above points are important only for alignment and star measurement, not for auroras, and for these purposes it would also be better to have an alternative higher magnification (50-100 per cent) than with the standard design. The power of the eye lens could be stretched to a point at which the aberrations of the eyepiece are still not too great; they matter less anyway in this incidental use.

The diameter of the parallactic graticule should be kept to a minimum, i.e. it should not be larger than the diameter of the measurable field, otherwise there is difficulty in adjusting the uniform close spacing between the graticule and the photographic plate.

The slides described in Part 1 providing a lateral motion of the theodolite require a very high grade of engineering. They were designed to provide a motion of 1 inch each way from a central position. Within half of that range they manifest no measurable change of the theodolite's pointing which means, allowing for only 1·7 times magnification and rather thick lines of the pointing cross, that an error of $\frac{1}{2}$ minute of arc would still be detected. Farther out they are less reliable and restraint had to be exerted in their use. Probably a non-kinematic design would be more reliable, or consideration should be given to locating the objective of the theodolite at its trunnions.

Finally reference is again made to Part 1 (Conclusions) for a possible wider scope of applications of the collimating principle.

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